QUANTIZATION OF LIE BIALGEBRAS AND SHUFFLE ALGEBRAS OF LIE ALGEBRAS

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ABSTRACT. To any field $\mathbb{K}$ of characteristic zero, we associate a set $\mathrm{III}(\mathbb{K})$. Elements of $\mathrm{III}(\mathbb{K})$ are equivalence classes of families of Lie polynomials subject to associativity relations. We construct an injection and a retraction between $\mathrm{III}(\mathbb{K})$ and the set of quantization functors of Lie bialgebras over $\mathbb{K}$. This construction involves the following steps. 1) To each element ϖ of $\mathrm{III}(\mathbb{K})$, we associate a functor $\mathfrak{a} \mapsto \mathrm{Sh}^{\varpi}(\mathfrak{a})$ from the category of Lie bialgebras to that of Hopf algebras; $\mathrm{Sh}^{\varpi}(\mathfrak{a})$ contains $U\mathfrak{a}$. 2) When $\mathfrak{a}$ and $\mathfrak{b}$ are Lie algebras, and $r_{\mathfrak{a}\mathfrak{b}} \in \mathfrak{a} \otimes \mathfrak{b}$, we construct an element $\mathcal{R}^{\varpi}(r_{\mathfrak{a}\mathfrak{b}})$ of $\mathrm{Sh}^{\varpi}(\mathfrak{a}) \otimes \mathrm{Sh}^{\varpi}(\mathfrak{b})$ satisfying quasitriangularity identities; in particular, $\mathcal{R}^{\varpi}(r_{\mathfrak{a}\mathfrak{b}})$ defines a Hopf algebra morphism from $\mathrm{Sh}^{\varpi}(\mathfrak{a})^{*}$ to $\mathrm{Sh}^{\varpi}(\mathfrak{b})$. 3) When $\mathfrak{a} = \mathfrak{b}$ and $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution of CYBE, we construct a series $\rho^{\varpi}(r_{\mathfrak{a}})$ such that $\mathcal{R}^{\varpi}(\rho^{\varpi}(r_{\mathfrak{a}}))$ is a solution of QYBE. The expression of $\rho^{\varpi}(r_{\mathfrak{a}})$ in terms of $r_{\mathfrak{a}}$ involves Lie polynomials, and we show that this expression is unique at a universal level. This step relies on vanishing statements for cohomologies arising from universal algebras for the solutions of CYBE. 4) We define the quantization of a Lie bialgebra $\mathfrak{g}$ as the image of the morphism defined by $\mathcal{R}^{\varpi}(\rho^{\varpi}(r))$, where $r \in \mathfrak{g} \otimes \mathfrak{g}^{*}$ is the canonical element attached to $\mathfrak{g}$.

Introduction. According to Drinfeld, a quantum group is a formal deformation of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. The semiclassical structure associated with such a deformation is a Lie bialgebra structure on $\mathfrak{g}$. The quantization problem of Lie bialgebras, as posed by Drinfeld in [5, 6], is to construct a functor from the category of Lie bialgebras to that of quantum groups, whose composition with the "semiclassical limit" functor is the identity. Such an object is called a quantization functor. When the structure constants of the quantum group can be expressed by polynomial formulas in terms of those of the Lie bialgebra, the quantization functor is called universal.

An approach to the construction of universal quantization functors was proposed by N. Reshetikhin ([16]). Later, it was solved by P. Etingof and D. Kazhdan ([8]). Their quantization procedure involves associators. An associator in an element of an abstract algebra, subject to certain conditions. An example of an associator is Drinfeld's "Knizhnik-Zamolodchikov associator", which involves special values of multiple zeta functions. Drinfeld also proved the existence of nontrivial associators defined over $\mathbb{Q}$. Let $\mathbb{K}$ be a field of characteristic zero, and let us denote by **Assoc**$(\mathbb{K})$ the set of associators defined over $\mathbb{K}$. The main result

of [8] is the construction of a map $EK : \mathbf{Assoc}(\mathbb{K}) \rightarrow \{$universal quantization functors of Lie bialgebras over $\mathbb{K}\}$.

Our purpose in this paper is to study further the set of quantization functors of Lie bialgebras. Our main result may be stated as follows. We introduce a set $\text{Ш}(\mathbb{K})$ of equivalence classes of Lie polynomials, satisfying certain associativity equations, and we define two maps $\alpha_{\mathbb{K}} : \text{Ш}(\mathbb{K}) \rightarrow \{$universal quantization functors of Lie bialgebras over $\mathbb{K}\}$ and $\beta_{\mathbb{K}} : \{$universal quantization functors of Lie bialgebras over $\mathbb{K}\} \rightarrow \text{Ш}(\mathbb{K})$, such that $\beta_{\mathbb{K}} \circ \alpha_{\mathbb{K}} = \text{id}_{\text{Ш}(\mathbb{K})}$. In particular, the composition $\beta_{\mathbb{K}} \circ EK$ yields a natural map $\mathbf{Assoc}(\mathbb{K}) \rightarrow \text{Ш}(\mathbb{K})$. The image of $\alpha_{\mathbb{K}}$ is contained in the set of universal quantization functors with functorial behavior with respect to the operations of dualizing and taking the double of Lie bialgebras. We set up a bijection between this set of quantization functors with the product of $\text{Ш}(\mathbb{K})$ with a universal group $\mathcal{G}_0$.

Our approach may be viewed as close to the original approach of Reshetikhin. To describe the latter, it is useful to recall how quantum Kac-Moody algebras were initially constructed in [4, 11].

Let $\mathfrak{a}$ be a Kac-Moody Lie algebra, and let $\delta : \mathfrak{a} \rightarrow \wedge^2 \mathfrak{a}$ be the cocycle defining its standard (Drinfeld-Sklyanin) Lie bialgebra structure. Then there exist two opposite Borel subalgebras $\mathfrak{b}_+$ and $\mathfrak{b}_-$ of $\mathfrak{a}$, which are Lie subbialgebras of $\mathfrak{a}$. The quantization of $\mathfrak{b}_\pm$ is constructed as follows. Let $h_i, x_i^\pm$ be the generators of $\mathfrak{b}_\pm$. Then δ is expressed simply on these generators, by $\delta(h_i) = 0$, $\delta(x_i^\pm) = \pm d_i h_i \wedge x_i^\pm$. One then defines the coproduct Δ on these generators by $\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i$, $\Delta(x_i^\pm) = x_i^\pm \otimes e^{\pm d_i h_i} + 1 \otimes x_i^\pm$; up to now the only condition on Δ is that it should be coassociative and it should deform δ. This means that we have defined Hopf algebra structures on the tensor algebras of $(\oplus_i \mathbb{C} h_i) \oplus (\oplus_i \mathbb{C} x_i^\pm)$. One then finds relations between the generators h_i and $x_i^\pm$, which deform the classical relations, and are skew-primitive with respect to Δ. One may then show that the resulting algebras are flat deformations of $U\mathfrak{b}_\pm$, and that the relations are the generators of the radical of a Hopf pairing between $T\big((\oplus_i \mathbb{C} h_i) \oplus (\oplus_i \mathbb{C} x_i^+)\big)$ and $T\big((\oplus_i \mathbb{C} h_i) \oplus (\oplus_i \mathbb{C} x_i^-)\big)$ (see e.g. [13]). $U_\hbar \mathfrak{a}$ is then constructed as a subalgebra of the quantum double of $U_\hbar \mathfrak{b}_+$, because this quantum double contains two copies of the Cartan subalgebra of $\mathfrak{a}$. The procedure is the same in the case of the quantum current algebras ("new realizations algebras").

The approach of Reshetikhin to the quantization problem was to imitate these steps in the general case. Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra and let $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ be the dual Lie bialgebra. Then the tensor algebras $T(\mathfrak{g})$ and $T(\mathfrak{g}^*)$ of $\mathfrak{g}$ and $\mathfrak{g}^*$ are cocommutative Hopf algebras, and $\delta_{\mathfrak{g}}$ and $\delta_{\mathfrak{g}^*}$ extend to Hopf co-Poisson structures on $T(\mathfrak{g})$ and $T(\mathfrak{g}^*)$. The first step is then to construct coproducts $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{g}^*}$ on these tensor algebras, deforming $\delta_{\mathfrak{g}}$ and $\delta_{\mathfrak{g}^*}$. The second step is to construct a Hopf algebra pairing between $(T(\mathfrak{g}), \Delta_{\mathfrak{g}})$ and $(T(\mathfrak{g}^*), \Delta_{\mathfrak{g}^*})$ and define the quantizations of $\mathfrak{g}$ and $\mathfrak{g}^*$ as the quotients of $T(\mathfrak{g})$ and $T(\mathfrak{g}^*)$ by the radicals of this

pairing. The pairing between $T(\mathfrak{g})$ and $T(\mathfrak{g}^*)$ should be chosen in such a way that the quotients are flat deformations of $U\mathfrak{g}$ and $U(\mathfrak{g}^*)$.

In this paper, we study this program placing ourselves in the dual framework. For any vector space V, the dual to the Hopf algebra structure on its tensor algebra $T(V)$ is a Hopf algebra structure defined on the shuffle algebra $\mathrm{Sh}(V^*)$ of its dual. The duals of $(T(\mathfrak{g}), \delta_\mathfrak{g})$ and $(T(\mathfrak{g}^*), \delta_{\mathfrak{g}^*})$ are then commutative Hopf-Poisson structures on $\mathrm{Sh}(\mathfrak{g}^*)$ and $\mathrm{Sh}(\mathfrak{g})$. Moreover, the map $\mathfrak{g} \mapsto \mathrm{Sh}(\mathfrak{g})$ is a functor from the category of Lie algebras to that of Hopf-Poisson algebras. The dual translation of the first step of the above program is to construct a functorial quantization of these Hopf-Poisson structures.

In Section 1, we introduce the main new object of this paper. This is a functor $\mathbb{K} \mapsto \text{Ш}(\mathbb{K})$ from the category of fields to that of sets. Elements of $\text{Ш}(\mathbb{K})$ are equivalence classes of families of Lie polynomials, subject to certain relations (see Section 1.1). We construct a bijection between $\text{Ш}(\mathbb{K})$ and the set of universal quantization functors of the Hopf-Poisson structure of $\mathrm{Sh}(\mathfrak{g})$ (Proposition 1.4). In Section 1.5, we explain the connection between the quantizations of $\mathrm{Sh}(\mathfrak{g})$ provided by elements of $\text{Ш}(\mathbb{K})$ and the PBW quantization of $S\mathfrak{g}$ ([3, 12]). (The shuffle algebras appeared many times in the theory of Hopf algebras and quantum groups (see [15, 1, 20, 19, 18]), but the construction of $\mathrm{Sh}(\mathfrak{g})$ using elements of $\text{Ш}(\mathbb{K})$ seems to be new.) If ϖ is an element of $\text{Ш}(\mathbb{K})$, we denote by $\mathfrak{g} \mapsto \mathrm{Sh}^\varpi(\mathfrak{g})$ the corresponding quantization functor.

When (A, m_A, Δ_A) and (B, m_B, Δ_B) are any finite-dimensional Hopf algebras, a Hopf pairing between them is the same as an element $\mathcal{R}_{AB}$ in $A^* \otimes B^*$, such that $(\Delta_{A^*} \otimes \mathrm{id})(\mathcal{R}_{AB}) = \mathcal{R}_{AB}^{(13)} \mathcal{R}_{AB}^{(23)}$ and $(\mathrm{id} \otimes \Delta_{B^*})(\mathcal{R}_{AB}) = \mathcal{R}_{AB}^{(12)} \mathcal{R}_{AB}^{(13)}$ (where Δ_{A^*} and Δ_{B^*} are the dual maps to the multiplication maps of A and B). In that case, the map $\ell : B^* \to A$ defined by $\ell(\xi) = \langle \mathcal{R}_{AB}, \mathrm{id} \otimes \xi \rangle$ defines a Hopf algebra morphism from B^* to A. With some restrictions, the same is true in our situation. Our next step is then to associate to any $\varpi \in \text{Ш}(\mathbb{K})$, any pair $(\mathfrak{a}, \mathfrak{b})$ of Lie algebras and any element $r_{\mathfrak{a}\mathfrak{b}}$ of $\mathfrak{a} \otimes \mathfrak{b}$, a family of elements $\mathcal{R}_n^\varpi(r_{\mathfrak{a}\mathfrak{b}})$ in $\mathrm{Sh}^\varpi(\mathfrak{a}) \otimes \mathrm{Sh}^\varpi(\mathfrak{b})$, such that if t is any formal parameter, $\mathcal{R}^\varpi(r_{\mathfrak{a}\mathfrak{b}}) = \sum_{n \geq 0} t^n \mathcal{R}_n^\varpi(r_{\mathfrak{a}\mathfrak{b}})$ satisfies these identities (Proposition 2.1).

In the rest of the construction, we fix an element ϖ in $\text{Ш}(\mathbb{K})$. Assume that $\mathfrak{a} = \mathfrak{b}$ and set $r_\mathfrak{a} = r_{\mathfrak{a}\mathfrak{b}}$. It is natural to ask when $\mathcal{R}^\varpi(r_\mathfrak{a})$ satisfies the quantum Yang-Baxter equation (QYBE). To state our next main result, we need some notation. Denote by FL_n the part of the free Lie algebra in n generators, homogeneous of degree one in each generator. The symmetric group $\mathfrak{S}_n$ acts by simultaneous permutations of generators of both factors of $FL_n \otimes FL_n$. We set $F_n = (FL_n \otimes FL_n)_{\mathfrak{S}_n}$. Then there is a unique linear map $\kappa_n^{(ab)}(r_\mathfrak{a}) : F_n \to \mathfrak{a} \otimes \mathfrak{a}$, sending the class of $P(x_1, \ldots, x_n) \otimes Q(x_1, \ldots, x_n)$ to $\sum_{i_1, \ldots, i_n \in I} P(a_{i_1}, \ldots, a_{i_n}) \otimes Q(b_{i_1}, \ldots, b_{i_n})$, if $r_\mathfrak{a}$ has the form $\sum_{i \in I} a_i \otimes b_i$. Our next main result is (see Theorem 3.4, Lemma 3.3, Corollary 3.1 and Proposition 3.1)

Theorem 0.1. *There exists a family $(\varrho_n^{\varpi})_{n \geq 1}$ in $\prod_{n \geq 1} F_n$, such that $\varrho_1^{\varpi} = x_1 \otimes x_1$, and if $\mathfrak{a}$ is Lie algebra and $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution of the classical Yang-Baxter equation (CYBE), then $\mathcal{R}^{\varpi}(\sum_{n \geq 1} \kappa_n^{(ab)}(r_{\mathfrak{a}})(\varrho_n^{\varpi}))$ is a solution of QYBE in $\mathrm{Sh}^{\varpi}(\mathfrak{a})^{\otimes 3}$.*

We find the family $(\varrho_n^{\varpi})_{n \geq 1}$ as the solution of a system of equations in $\prod_{n \geq 1} F_n$, the *universal Lie QYB equations* (see Section 3.2.2). More precisely, we prove that the family $(\varrho_n^{\varpi})_{n \geq 1}$ is the unique solution to these equations (see Theorem 3.4).

For any ϖ, we have $\varrho_2^{\varpi} = \frac{1}{8}[x_1, x_2] \otimes [x_1, x_2]$, so that the expansion of $\rho^{\varpi}(r_{\mathfrak{a}}) = \sum_{n \geq 1} \kappa_n^{(ab)}(r_{\mathfrak{a}})(\varrho_n^{\varpi})$ is

$$\rho^{\varpi}(r_{\mathfrak{a}}) = \sum_i a_i \otimes b_i + \frac{1}{8} \sum_{i,j \in I} [a_i, a_j] \otimes [b_i, b_j] + \cdots .$$

The universal Lie QYB equations are equalities in an algebra $F^{(3)}$, which is a particular instance of a family of algebras $F^{(N)}$ (see Appendices B and C). The algebras $F^{(N)}$ are universal algebras for the pairs $(\mathfrak{a}, r_{\mathfrak{a}})$ of a Lie algebra $\mathfrak{a}$ and a solution $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ of CYBE. When A is any algebra, we connect the problem of deforming a solution $r_A \in A \otimes A$ of CYBE into a solution of QYBE with the Lie coalgebra structure on A defined by r_A, more precisely, with the corresponding Lie coalgebra cohomology (Proposition 3.5; the complete statement is in Theorem A.1 of Appendix A). Adapting this result to a family of universal shuffle algebras $\mathrm{Sh}_k^{(F)}$ (see Theorem 3.2 and Appendix E), we formulate the universal Lie QYB equations in terms of cohomology groups H_n^2 and H_n^3, constructed in terms of the family $(F^{(N)})_{N=2,3,4}$. We then show vanishing statements for the groups H_n^2 and H_n^3 (Theorem 3.3 and Appendix D), using identities in free Lie algebras (Propositions D.1 and D.2, see also [17]) . This shows the existence and unicity of the solution $(\varrho_n^{\varpi})_{n \geq 1}$ to the universal Lie QYB equations (Theorem 3.4).

We then apply this construction to the problem of universal quantization of Lie bialgebras (Section 4). To a finite-dimensional Lie bialgebra $\mathfrak{g}$ over $\mathbb{K}$ are attached its dual Lie bialgebra $\mathfrak{g}^*$, its double $\mathfrak{d}$, and the canonical r-matrix $r_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^*$ (see Section 4.1 and [5]). $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{d}$, so $r_{\mathfrak{g}} \in \mathfrak{d} \otimes \mathfrak{d}$; $r_{\mathfrak{g}}$ is then a solution of CYBE. Then

$$\mathcal{R}^{\varpi}(\rho^{\varpi}(\hbar r_{\mathfrak{g}})) \in \mathrm{Sh}^{\varpi}(\mathfrak{g}) \otimes \mathrm{Sh}^{\varpi}(\mathfrak{g}^*)[[\hbar]] \subset \mathrm{Sh}^{\varpi}(\mathfrak{d})^{\otimes 2}[[\hbar]],$$

and as an element of the latter algebra, $\mathcal{R}^{\varpi}(\rho(\hbar r_{\mathfrak{g}}))$ is a solution of QYBE (here $\hbar$ is a formal parameter). As we explained above, $\mathcal{R}^{\varpi}(\rho^{\varpi}(\hbar r_{\mathfrak{g}}))$ induces a Hopf algebra morphism ℓ from a Hopf algebra $T_{\hbar}^{\varpi}(\mathfrak{g})$ dual to $\mathrm{Sh}^{\varpi}(\mathfrak{g}^*)$ to $\mathrm{Sh}^{\varpi}(\mathfrak{g})[[\hbar]]$. We then define $U_{\hbar}^{\varpi}\mathfrak{g}$ as the image $\mathrm{Im}(\ell)$ of ℓ and show that is a quantization of the Lie bialgebra $\mathfrak{g}$ (Theorem 4.1). We also prove that $\mathrm{Im}(\ell)$ is divisible in $\mathrm{Sh}^{\varpi}(\mathfrak{g})[[\hbar]]$ (as a $\mathbb{K}[[\hbar]]$-module). The fact that $\mathcal{R}^{\varpi}(\rho^{\varpi}(\hbar r_{\mathfrak{g}}))$ is a solution of QYBE plays an essential role in the proof of both results; the idea from [7] that the subalgebras of

shuffle algebras are necessarily torsion-free is also at the basis of the construction. We show that the map $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g}, \delta_\mathfrak{g}) \mapsto U_\hbar^\varpi \mathfrak{g}$ defines a functor from the category of Lie bialgebras to that of Hopf algebras (Proposition 4.1). We comment on the extension of this functor to infinite-dimensional Lie bialgebras in Remark 8.

We also characterize the quantized formal series Hopf (QFSH) algebra $\mathcal{O}_\hbar^\varpi(G^*)$ corresponding to $U_\hbar^\varpi \mathfrak{g}$ as the intersection of $\mathrm{Im}(\ell)$ with a subalgebra $\mathrm{Sh}_\hbar^\varpi(\mathfrak{g})$ of $\mathrm{Sh}^\varpi(\mathfrak{g})[[\hbar]]$ (Proposition 4.2). According to a general result ([5, 10]), $\mathcal{O}_\hbar^\varpi(G^*)$ is a quantization of the Hopf algebra of functions on the formal Poisson-Lie group corresponding to $\mathfrak{g}^*$; we show that $\mathcal{O}_\hbar^\varpi(G^*)$ coincides with the dual $(U_\hbar^\varpi \mathfrak{g}^*)^*$ of $U_\hbar^\varpi \mathfrak{g}^*$.

According to Theorem 4.1 and Proposition 4.1, there is a unique map $\alpha_\mathbb{K}$: $\text{Ш}(\mathbb{K}) \to \{$universal quantization functors of Lie bialgebras over $\mathbb{K}\}$ such that $\alpha_\mathbb{K}(\varpi) = (\mathfrak{g} \mapsto U_\hbar^\varpi(\mathfrak{g}))$. On the other hand, we define a map $\beta_\mathbb{K}$: $\{$universal quantization functors of Lie bialgebras over $\mathbb{K}\} \to \text{Ш}(\mathbb{K})$ as follows. When V is a vector space, denote by $F(V)$ the free Lie algebra of V. If $\mathfrak{g}$ is a Lie algebra, then $F(\mathfrak{g}^*)$ is naturally equipped with a structure of Lie bialgebra. One shows that the restriction of a quantization functor Q to bialgebras of the form $F(\mathfrak{g}^*)$ yields an element $\beta_\mathbb{K}(Q)$ of $\text{Ш}(\mathbb{K})$.

If $\mathfrak{g} \mapsto Q(\mathfrak{g})$ is a universal quantization functor for Lie bialgebras, let us say that it is compatible with the operations of dual and double if the following holds: if $\mathfrak{g}$ is any Lie bialgebra, then there is are canonical isomorphisms between $Q(\mathfrak{g}^*)$ and $(Q(\mathfrak{g})^*)^\vee$ ($O^\vee$ is the quantized universal enveloping algebra attached to any quantized formal series Hopf algebra O, see [5, 10]) and between $Q(D(\mathfrak{g}))$ and the quantum double of $Q(\mathfrak{g})$ ($D(\mathfrak{g})$ is the double of a Lie algebra $\mathfrak{g}$). In that case, we will say that Q is a compatible functor.

Let us denote by $\mathcal{G}_0$ the group of all functorial assignments $\mathfrak{a} \mapsto \rho_\mathfrak{a}$, where $\mathfrak{a}$ is a Lie bialgebra and $\rho_\mathfrak{a} \in \mathrm{End}(\mathfrak{a})[[\hbar]]$, such that $\rho_\mathfrak{a}$ is an iterate of tensor products of the Lie bracket and cobracket, and satisfies the identities $\rho_\mathfrak{a}([x,y]) = [\rho_\mathfrak{a}(x), y]$, $\rho_{\mathfrak{a}^*} = \rho_\mathfrak{a}^t$ and $\rho_\mathfrak{a} = \mathrm{id}_\mathfrak{a} + o(\hbar)$. Then $\mathcal{G}_0$ may be viewed as a group of universal transformations of the solutions of CYBE arising from Lie bialgebras (see Lemma 5.1 and Remark 9), and as a "linear" subgroup of the group of functorial assignments $(\mathfrak{a} \mapsto \rho_\mathfrak{a})$, such that if $[\ ,\ ]_\mathfrak{a}$ and $\delta_\mathfrak{a}$ are the Lie bracket and cobracket of $\mathfrak{a}$, then $(\mathfrak{a}, [\ ,\ ]_\mathfrak{a}, (\rho_\mathfrak{a} \otimes \rho_\mathfrak{a}) \circ \delta_\mathfrak{a} \circ \rho_\mathfrak{a}^{-1})$ is a Lie bialgebra.

In Section 5, we prove

Theorem 0.2. *The map $\alpha_\mathbb{K}$ is an injection and $\beta_\mathbb{K}$ is a retraction of $\alpha_\mathbb{K}$, that is $\beta_\mathbb{K} \circ \alpha_\mathbb{K} = \mathrm{id}_{\text{Ш}(\mathbb{K})}$. These maps depend functorially on $\mathbb{K}$ and are equivariant with respect to the natural actions of $\mathrm{Aut}(\mathbb{K}[[\hbar]])$ on $\text{Ш}(\mathbb{K})$ and $\{$universal quantization functors of Lie bialgebras over $\mathbb{K}\}$.*

We have $\alpha_\mathbb{K}(\text{Ш}(\mathbb{K})) \subset \{$compatible quantization functors for Lie bialgebras$\}$. There is a bijection $\varepsilon_\mathbb{K}$ between $\text{Ш}(\mathbb{K}) \times \mathcal{G}_0$ and $\{$compatible quantization functors for Lie bialgebras$\}$, such that $\beta_\mathbb{K} \circ \varepsilon_\mathbb{K}$ is the projection on the first factor and the restriction of $\varepsilon_\mathbb{K}$ on $\text{Ш}(\mathbb{K}) \times \{$neutral element$\}$ coincides with $\alpha_\mathbb{K}$.

In particular, if $\mathcal{G}_0$ reduces to multiplication by scalars, then $\alpha_{\mathbb{K}}$ sets up a bijection between $\text{Ш}(\mathbb{K})$ and {compatible quantization functors for Lie bialgebras}.

The proof of this Theorem is in Section 5. It relies on the following idea. If $\mathfrak{g}$ is a Lie algebra, then the universal enveloping algebra of $F(\mathfrak{g})$ is the Hopf-co-Poisson algebra $(T(\mathfrak{g}), \delta_{\mathfrak{g}})$. The first part of the proof of Theorem 0.2 involves the fact that the quantizations of $(T(\mathfrak{g}), \delta_{\mathfrak{g}})$ given by $\text{Sh}^{\varpi}(\mathfrak{g})^*$ and $U_{\hbar}^{\varpi}(F(\mathfrak{g}^*))$ are naturally isomorphic (Section 5.1). The second part of the proof relies on the fact that any Lie bialgebra $\mathfrak{g}$ may be viewed as the image of a Lie bialgebra morphism from $F(\mathfrak{g})$ to $F(\mathfrak{g}^*)^*$.

The map $\textbf{Assoc}(\mathbb{K}) \to \text{Ш}(\mathbb{K})$ may help to study how the quantization functors of Etingof and Kazhdan depend on associators. More precisely, the image of the map EK is contained in {compatible quantization functors for Lie bialgebras}, so if $\mathcal{G}_0$ is trivial, then the map $EK(\textbf{Assoc}(\mathbb{K})) \to \text{Ш}(\mathbb{K})$ is an injection.

For some classes of Lie algebras, the Etingof-Kazhdan quantizations are known to be independent of the associator (for example in the case of Kac-Moody algebras (see [9]) and nondegenerate triangular Lie bialgebras ([14])).

1. $\mathrm{III}(\mathbb{K})$ AND SHUFFLE ALGEBRAS OF LIE ALGEBRAS

1.1. **Definition of** $\mathrm{III}(\mathbb{K})$. Let R be a commutative ring. Let us define $FL_n(R)$ as the part of the free Lie algebra over R with n generators, homogeneous of degree 1 in each generator. If λ belongs to R, let us define $\mathcal{B}(R)_\lambda$ as the set of all families $(B_{pq})_{p,q\geq 0}$, where each B_{pq} belongs to $FL_{p+q}(R)$, such that $(B_{pq})_{p,q\geq 0}$ satisfies the equations

$$B_{10}(x) = B_{01}(x) = x, \quad B_{p0} = B_{0p} = 0 \text{ if } p \neq 1, \quad B_{11}(x_1, x_2) = \lambda[x_1, x_2],$$

and

$$\sum_{\alpha>0} \sum_{(p_\beta)_{\beta=1,\dots,\alpha}\in\mathrm{Part}_\alpha(p),(q_\beta)_{\beta=1,\dots,\alpha}\in\mathrm{Part}_\alpha(q)} B_{\alpha r}\big(B_{p_1 q_1}(x_1,\dots,x_{p_1}|y_1,\dots,y_{q_1})\cdots \tag{1}$$

$$\cdots B_{p_\alpha q_\alpha}(x_{\sum_{\beta=1}^{\alpha-1} p_\beta+1},\dots,x_p|y_{\sum_{\beta=1}^{\alpha-1} q_\beta+1},\dots,y_q)|z_1,\dots,z_r\big)$$

$$= \sum_{\alpha>0} \sum_{(q_\beta)_{\beta=1,\dots,\alpha}\in\mathrm{Part}_\alpha(q),(r_\beta)_{\beta=1,\dots,\alpha}\in\mathrm{Part}_\alpha(r)} B_{p\alpha}\big(x_1,\dots,x_p|$$

$$B_{q_1 r_1}(y_1,\dots,y_{q_1}|z_1,\dots,z_{r_1})\cdots B_{q_\alpha r_\alpha}(y_{\sum_{\beta=1}^{\alpha-1} q_\beta+1},\dots,y_q|z_{\sum_{\beta=1}^{\alpha-1} r_\beta+1},\dots,z_r)\big)$$

for any integers $p, q, r > 0$. Here $\mathrm{Part}_\alpha(n)$ is the set of α-partitions of n, that is the set of αuples $(n_1,\dots,n_\alpha)$ of nonnegative integers such that $\sum_{\beta=1}^{\alpha} n_\beta = n$. We also set $\mathcal{B}(R) = \coprod_{\lambda\in R} \mathcal{B}(R)_\lambda$.

Let us define $\mathcal{G}(R)$ as the subset of the product $\prod_{n\geq 1} FL_n(R)$ of elements $(P_n)_{n\geq 1}$ such that $P_1(x) = x$. If $P = (P_n)_{n\geq 1}$ and $Q = (Q_n)_{n\geq 1}$ belong to $\mathcal{G}(R)$, let us define $(P * Q)_n$ as the element of $FL_n(R)$ equal to

$$(P * Q)(x_1,\dots,x_n) = \sum_{\alpha>0} \sum_{(n_1,\dots,n_\alpha)\in\mathrm{Part}_\alpha(n)} P_\alpha(Q_{n_1}(x_1,\dots,x_{n_1}),\dots, \tag{2}$$

$$Q_{n_\alpha}(x_{\sum_{\alpha'=1}^{\alpha} n_{\alpha'}+1},\dots,x_n)).$$

Then $*$ defines a group structure on $\mathcal{G}(R)$.

If $P = (P_n)_{n\geq 1}$ belongs to $\mathcal{G}(R)$ and $B = (B_{pq})_{p,q\geq 0}$ belongs to $\mathcal{B}(R)$, let us define $(P * B)_{pq}$ as the element of FL_{p+q} defined by

$$(P * B)_{pq}(x_1,\dots,x_p|y_1,\dots,y_q) = \sum_{\alpha,\beta>0} \sum_{(p_1,\dots,p_\alpha)\in\mathrm{Part}_\alpha(p),(q_1,\dots,q_\beta)\in\mathrm{Part}_\beta(q)}$$

$$B_{\alpha\beta}\big(P_{p_1}(x_1,\dots,x_{p_1}),\dots,P_{p_\alpha}(x_{\sum_{\alpha'=1}^{\alpha-1} p_{\alpha'}+1},\dots,x_p)|$$

$$P_{q_1}(y_1,\dots,y_{q_1}),\dots,P_{q_\alpha}(y_{\sum_{\beta'=1}^{\beta-1} q_{\alpha'}+1},\dots,y_q)\big).$$

Then $*$ defines an action of $\mathcal{G}(R)$ on each $\mathcal{B}(R)_\lambda$ and on $\mathcal{R}(R)$.

We define $\widetilde{\mathrm{III}}(R)$ and $\widetilde{\mathrm{III}}(R)_\lambda$ as the quotient sets $\mathcal{B}(R)/\mathcal{G}(R)$ and $\mathcal{B}(R)_\lambda/\mathcal{G}(R)$. If $\mathbb{K}$ is a field of characteristic zero, we set $\mathrm{III}(\mathbb{K}) = \widetilde{\mathrm{III}}(\mathbb{K}[[\hbar]])$, where $\hbar$ is a formal variable. We also set $\mathrm{III}(\mathbb{K})_\lambda = \widetilde{\mathrm{III}}(\mathbb{K}[[\hbar]])_\lambda$, if $\lambda \in \mathbb{K}[[\hbar]]$.

If R is a ring, then there is a unique involution $\omega \mapsto \omega^\vee$ of $\mathcal{B}(R)$, such that if $\omega = (B_{pq})_{p,q \geq 0}$, then $\omega^\vee = (B_{pq}^\vee)_{p,q \geq 0}$, where $B_{pq}^\vee(x_1, \ldots, x_p | y_1, \ldots, y_q) = B_{pq}(x_p, \ldots, x_1 | y_q, \ldots, y_1)$. The map $\omega \mapsto \omega^\vee$ induces an involution $\varpi \mapsto \varpi^\vee$ of $\widetilde{\mathrm{III}}(R)$ and of $\mathrm{III}(\mathbb{K})$, if $\mathbb{K}$ is any field.

The group $R^\times$ also acts on $\mathcal{B}(R)$ by the rule $r \cdot (B_{pq})_{p,q} = (r^{p+q-1} B_{pq})_{p,q}$. This action induces an action of $R^\times$ on $\widetilde{\mathrm{III}}(R)$. The group $\mathrm{Aut}(R)$ also acts on $\widetilde{\mathrm{III}}(R)$ in the obvious way.

Remark 1. Define $\mathcal{B}_n(R)$ as the set of families $(B_{pq})_{p,q | p,q \geq 0, p+q \leq n}$ satisfying the relations (1), for any p, q, r such that $p + q + r \leq n$. Define $\mathcal{G}_n(R)$ as the set of families $(P_k)_{k | 1 \leq k \leq n}$. Then the rule (2) equips $\mathcal{G}_n(R)$ with a group structure. Moreover, there are natural projection maps $\mathcal{B}_{n+1}(R) \to \mathcal{B}_n(R)$ and $\mathcal{G}_{n+1}(R) \to \mathcal{G}_n(R)$, which induce maps $\widetilde{\mathrm{III}}_{n+1}(R) \to \widetilde{\mathrm{III}}_n(R)$. Then $\widetilde{\mathrm{III}}(R)$ is the projective limit $\lim_{\leftarrow n} \widetilde{\mathrm{III}}_n(R)$.

1.2. **Hopf-Poisson structures on shuffle algebras.**

Let $\mathbb{K}$ be a field of characteristic zero and let V be a vector space over $\mathbb{K}$. The shuffle algebra $\mathrm{Sh}(V)$ is a commutative Hopf algebra; it is defined as follows. As a vector space, $\mathrm{Sh}(V)$ is isomorphic with the tensor algebra $T(V)$. If $v_1, \ldots, v_n$ are elements of V, we denote by $(v_1 \ldots v_n)$ the element of $\mathrm{Sh}(V)$ corresponding to $v_1 \otimes \cdots \otimes v_n$ by this isomorphism.

If p and q are integers, let us denote by $\mathfrak{S}_{p,q}$ the subset of $\mathfrak{S}_{p+q}$ of all permutations σ such that if (i, j) is a pair of integers such that $1 \leq i < j \leq p$ or $p + 1 \leq i < j \leq p + q$, then $\sigma(i) < \sigma(j)$ (shuffle permutations). The space $\mathrm{Sh}(V)$ is equipped with a multiplication m_0 defined by

$$m_0((v_1 \ldots v_n) \otimes (v_{n+1} \ldots v_{n+m})) = \sum_{\sigma \in \mathfrak{S}_{n,m}} (v_{\sigma(1)} \ldots v_{\sigma(n+m)})$$

and a comutiplication $\Delta_{\mathrm{Sh}(V)}$ defined by

$$\Delta_{\mathrm{Sh}(V)}((v_1 \ldots v_n)) = \sum_{k=0}^{n} (v_1 \ldots v_k) \otimes (v_{k+1} \ldots v_n).$$

If we define the unit of $\mathrm{Sh}(V)$ as $1 \in T^0(V)$, its counit $\varepsilon_{\mathrm{Sh}(V)}$ as the projection map on $T^0(V) = \mathbb{K}$ parallel to the sum $\oplus_{n>0} T^n(V)$, and the antipode by the formula $S_{\mathrm{Sh}(V)}((v_1 \ldots v_p)) = (-1)^p (v_p \ldots v_1)$, then $(\mathrm{Sh}(V), m_0, \Delta_{\mathrm{Sh}(V)}, \varepsilon_{\mathrm{Sh}(V)}, S_{\mathrm{Sh}(V)})$ is a commutative Hopf algebra.

Assume that V is the underlying vector space of a Lie algebra $\mathfrak{g}$. Then there is a unique linear map $m_1 : \mathrm{Sh}(\mathfrak{g})^{\otimes 2} \to \mathrm{Sh}(\mathfrak{g})$, such that

$$m_1((x_1 \ldots x_n) \otimes (x_{n+1} \ldots x_{n+m}))$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{\sigma' \in \mathfrak{S}_{i-1,j-1}, \sigma'' \in \mathfrak{S}_{n-i,m-j}} (y_{\sigma'(1)}^{(i,j)} \ldots y_{\sigma'(i+j-2)}^{(i,j)} [x_i, x_{n+j}] z_{\sigma''(1)}^{(i,j)} \ldots z_{\sigma''(n+m-i-j)}^{(i,j)})$$

for any $x_1, \ldots, x_{n+m}$ in $\mathfrak{g}$, where we set

$$(y_1^{(i,j)}, \ldots, y_{i+j-2}^{(i,j)}) = (x_1, \ldots, x_{i-1}, x_{n+1}, \ldots, x_{n+j-1}),$$

$(z_1^{(i,j)}, \ldots, z_{i+j-2}^{(i,j)}) = (x_{i+1}, \ldots, x_n, x_{n+j+1}, \ldots, x_{n+m})$ and $\mathfrak{S}_{i,j}$ is the set of shuffle permutations of $\mathfrak{S}_{i+j}$ associated to the partition $\{1, \ldots, i+j\} = \{1, \ldots, i\} \cup \{i+1, \ldots, i+j\}$.

The map m_1 defines a Poisson structure on the commutative algebra $\mathrm{Sh}(\mathfrak{g})$. Moreover, this Poisson structure is compatible with $\Delta_{\mathrm{Sh}(\mathfrak{g})}$, so $(\mathrm{Sh}(\mathfrak{g}), m_0, m_1, \Delta_{\mathrm{Sh}(\mathfrak{g})}, \varepsilon_{\mathrm{Sh}(\mathfrak{g})}, S_{\mathrm{Sh}(\mathfrak{g})})$ is a commutative Hopf-Poisson algebra; moreover, the map $\mathfrak{g} \mapsto (\mathrm{Sh}(\mathfrak{g}), m_0, m_1, \Delta_{\mathrm{Sh}(\mathfrak{g})})$ defines in a natural way a functor from the category of Lie algebras to that of Hopf-Poisson algebras.

In the same way, the tensor algebra $T(V)$ is equipped with a unique Hopf structure $(T(V), m_{T(V)}, \Delta_{T(V)}, \varepsilon_{T(V)}, S_{T(V)})$ such that the elements of $T^1(V)$ are primitive. If V is the underlying space of a Lie coalgebra $(\mathfrak{h}, \delta_{\mathfrak{h}})$, then there exists a unique co-Poisson map $\delta_{T(\mathfrak{h})} : T(\mathfrak{h}) \to T(\mathfrak{h})^{\otimes 2}$, whose restriction to $T^1(\mathfrak{h})$ coincides with $\delta_{\mathfrak{h}}$. $(T(\mathfrak{h}), m_{T(\mathfrak{h})}, \Delta_{T(\mathfrak{h})}, \delta_{T(\mathfrak{h})})$ is then a Hopf-co-Poisson algebra, and the map $\mathfrak{h} \mapsto (T(\mathfrak{h}), m_{T(\mathfrak{h})}, \Delta_{T(\mathfrak{h})}, \delta_{T(\mathfrak{h})})$ is a functor from the category of Lie coalgebras to that of Hopf co-Poisson algebras.

1.3. Quantizations of shuffle algebras.
Let $\omega = (B_{pq})_{p,q \geq 0}$ belong to $\mathcal{B}(\mathbb{K}[[\hbar]])$. If $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$, define $\mathrm{Sh}^\omega(\mathfrak{g})$ as follows. As a vector space, $\mathrm{Sh}^\omega(\mathfrak{g})$ is isomorphic to $T(\mathfrak{g})[[\hbar]]$. If $x_1, \ldots, x_n$ belong to $\mathfrak{g}$, let us denote by $(x_1 \ldots x_n)$ the element corresponding to $x_1 \otimes \cdots \otimes x_n$. Define a map $m_{\mathrm{Sh}^\omega(\mathfrak{g})} : \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} 2} \to \mathrm{Sh}^\omega(\mathfrak{g})$ by the rule

$$m_{\mathrm{Sh}^\omega(\mathfrak{g})}\big((x_1 \ldots x_n), (y_1 \ldots y_m)\big) = \sum_{k \geq 0} \sum_{(p_1 \ldots p_k) \text{ partition of } n, (q_1 \ldots q_k) \text{ partition of } m} \quad (3)$$

$$\big(B_{p_1 q_1}(x_1 \ldots x_{p_1} | y_1 \ldots y_{q_1}) \ldots B_{p_k q_k}(x_{p_1 + \cdots + p_{k-1} + 1} \ldots x_p | y_{q_1 + \cdots + q_{k-1} + 1} \ldots y_q)\big)$$

for any $x_1, \ldots, y_m$ in $\mathfrak{g}$ ($\widehat{\otimes}$ is the $\hbar$-adically completed tensor product) and $\Delta_{\mathrm{Sh}^\omega(\mathfrak{g})}$ as the unique linear map from $\mathrm{Sh}^\omega(\mathfrak{g})$ to $\mathrm{Sh}^\omega(\mathfrak{g}) \widehat{\otimes} \mathrm{Sh}^\omega(\mathfrak{g})$ such that

$$\Delta_{\mathrm{Sh}^\omega(\mathfrak{g})}((x_1 \ldots x_n)) = \sum_{i=0}^{n} (x_1 \ldots x_i) \otimes (x_{i+1} \ldots x_n),$$

for any $x_1, \ldots, x_n$ in $\mathfrak{g}$. Let us define $\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{g})}$ as the unique linear map from $\mathrm{Sh}^\omega(\mathfrak{g})$ to $\mathbb{K}$, such that the restriction of $\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{g})}$ to $\mathfrak{g}^{\otimes 0} = \mathbb{K}$ is the identity map, and the restriction of $\varepsilon_{\mathrm{Sh}(\mathfrak{g})}$ to $\oplus_{i>0} \mathfrak{g}^{\otimes i}$ is zero.

Finally, let us define inductively $S_{\mathrm{Sh}^\omega(\mathfrak{g})}$ as the unique endomorphism of $\mathrm{Sh}^\omega(\mathfrak{g})$ such that

$$S_{\mathrm{Sh}^\omega(\mathfrak{g})}(1) = 1, \quad S_{\mathrm{Sh}^\omega(\mathfrak{g})}((x_1 \ldots x_n)) = -\sum_{i=0}^{n-1} S_{\mathrm{Sh}^\omega(\mathfrak{g})}((x_1 \ldots x_i))(x_{i+1} \ldots x_n)$$

for any $x_1, \ldots, x_n$ in $\mathfrak{g}$. One checks that

$$S_{\mathrm{Sh}^\omega(\mathfrak{g})}(x_1 \ldots x_n) = \sum_{k \geq 0}(-1)^k \sum_{(p_1, \ldots, p_k) \text{ partition of } n} (x_1 \ldots x_{p_1}) \cdots (x_{p_1+\cdots+p_{k-1}+1} \ldots x_{p_1+\cdots+p_k}).$$

Proposition 1.1. $(\mathrm{Sh}^\omega(\mathfrak{g}), m_{\mathrm{Sh}^\omega(\mathfrak{g})}, \Delta_{\mathrm{Sh}^\omega(\mathfrak{g})}, \varepsilon_{\mathrm{Sh}^\omega(\mathfrak{g})}, S_{\mathrm{Sh}^\omega(\mathfrak{g})})$ *is a Hopf algebra. If* ω *belongs to* $\mathcal{B}(\mathbb{K})_{1/2}$, *then the subspace of* $\mathrm{Sh}^\omega(\mathfrak{g})$ *of all symmetric tensors is a Hopf subalgebra of* $\mathrm{Sh}^\omega(\mathfrak{g})$, *canonically isomorphic with* $U\mathfrak{g}$. *Moreover, the mapping* $\mathfrak{g} \mapsto \mathrm{Sh}^\omega(\mathfrak{g})$ *defines a functor from the category of Lie algebras to that of Hopf algebras.*

If ω *and* ω' *belong to the same orbit of the action of* $\mathcal{G}(\mathbb{K}[[\hbar]])$ *on* $\mathcal{B}(\mathbb{K}[[\hbar]])$, *then the Hopf algebras* $\mathrm{Sh}_\omega(\mathfrak{g})$ *and* $\mathrm{Sh}_{\omega'}(\mathfrak{g})$ *are canonically isomorphic (that is, the isomorphisms are depend functorially on* $\mathfrak{g}$). *If* ϖ *belongs to* $\mathrm{III}(\mathbb{K})$, *we will denote by* $\mathrm{Sh}^\varpi(\mathfrak{g})$ *any of the Hopf algebras* $\mathrm{Sh}^\omega(\mathfrak{g})$, *if* ϖ *is the coset of* ω.

Proof. $\Delta_{\mathrm{Sh}^\omega(\mathfrak{g})}$ is obviously coassociative. Moreover, it is also clear that $m_{\mathrm{Sh}^\omega(\mathfrak{g})}$ is a coalgebra map. The associativity of $m_{\mathrm{Sh}^\omega(\mathfrak{g})}$ follows from the identities (1). The fact that if x and y belong to $\mathrm{Sh}(\mathfrak{g})$, then $S_{\mathrm{Sh}(\mathfrak{g})}(xy) = S_{\mathrm{Sh}(\mathfrak{g})}(x)S_{\mathrm{Sh}(\mathfrak{g})}(y)$ is proved by induction on the degree of x any y. The other Hopf algebra axioms are checked directly. This proves the first statement.

The form of the product implies that the symmetric tensors form a subalgebra of $\mathrm{Sh}^\omega(\mathfrak{g})$. Let ι be the algebra morphism from $T(\mathfrak{g})$ to $\mathrm{Sh}^\omega(\mathfrak{g})$ such that for any x in $\mathfrak{g}$, $\iota(x) = (x)$. Then $\iota(x \otimes y - y \otimes y - [x, y])$ is equal to zero, so ι induces an algebra morphism $\bar{\iota}$ from $U\mathfrak{g}$ to the subalgebra of summetric elements of $\mathrm{Sh}^\omega(\mathfrak{g})$. The associated graded of this morphism coincides with the identity map of $S(\mathfrak{g})$, therefore $\bar{\iota}$ is an isomorphism.

Assume that $\omega = P * \omega'$, where $P = (P_i)_{i \geq 1} \in \mathcal{G}(\mathbb{K}[[\hbar]])$, then the canonical isomorphism $i_{\omega,\omega'}$ from $\mathrm{Sh}^\omega(\mathfrak{g})$ to $\mathrm{Sh}^{\omega'}(\mathfrak{g})$ is given by by the rule

$$i_{\omega,\omega'}(x_1 \ldots \ldots x_n) = \sum_{\alpha > 0} \sum_{(n_1, \ldots, n_\alpha) \in \mathrm{Part}_\alpha(n)} \left(P_{n_1}(x_1, \ldots, x_{n_1}) \cdots P_{n_\alpha}(x_{\sum_{\alpha'=1}^{\alpha-1} n_{\alpha'}+1} \ldots x_n)\right).$$

$\square$

Let us explain why we consider $\mathrm{Sh}^\omega(\mathfrak{g})$ as a quantization of the Hopf-Poisson algebra $(\mathrm{Sh}(\mathfrak{g}), m_0, m_1, \Delta_{\mathrm{Sh}(\mathfrak{g})}, \varepsilon_{\mathrm{Sh}(\mathfrak{g})}, S_{\mathrm{Sh}(\mathfrak{g})})$. Let us denote by $\mathrm{Sh}^{\omega, \leq i}(\mathfrak{g})$ the subspace of $\mathrm{Sh}^\omega(\mathfrak{g})$ equal to $\oplus_{i' \leq i} T^{i'}(\mathfrak{g})[[\hbar]]$. Then the sequence of inclusions $\mathrm{Sh}^{\omega, \leq 0}(\mathfrak{g}) \subset \mathrm{Sh}^{\omega, \leq 1}(\mathfrak{g}) \subset \cdots$ is a Hopf algebra filtration of $\mathrm{Sh}^\omega(\mathfrak{g})$. The associated graded algebra is commutative and inherits therefore a Poisson bracket. If ω belongs to $\mathcal{B}(\mathbb{K}[[\hbar]])_\lambda$, where $\lambda \in 1/2 + o(\hbar)$, the reduction modulo $\hbar$ of this Poisson algebra is $(\mathrm{Sh}(\mathfrak{g}), m_0, m_1)$.

1.4. $\mathrm{III}(\mathbb{K})$ **and quantizations of tensor algebras.** Let $(\mathfrak{a}, \delta_\mathfrak{a})$ be a Lie coalgebra. Let us consider the following quantization problem: to construct a topologically free $\mathbb{K}[[\hbar]]$-Hopf algebra, $(T, m_T, \Delta, \varepsilon_T, S_T)$ quantizing the Hopf-co-Poisson

structure $(T(\mathfrak{a}), m_{T(\mathfrak{a})}, \Delta^0_{T(\mathfrak{a})}, \delta_{T(\mathfrak{a})}, \varepsilon_{T(\mathfrak{a})}, S_{T(\mathfrak{a})})$ (we call $(T, m_T, \Delta_T, \varepsilon_T, S_T)$ a quantized tensor algebra).

The corresponding functorial problem is to construct all functors from the category of Lie coalgebras to that of quantized tensor algebras with the suitable classical limit. Let us say that such a functor is universal if the structure constants of the quantization of $T(\mathfrak{a})$ depends polynomially in those of $\mathfrak{a}$. The purpose of this Section is to construct a bijection between $\text{III}(\mathbb{K})$ and {universal quantization functors of the tensor algebras $T(\mathfrak{g})$}.

Proposition 1.2. *Let $(\mathfrak{a}, \delta_{\mathfrak{a}})$ be a Lie coalgebra, and let $(T, m_T, \Delta_T, \varepsilon_T, S_T)$ be a quantization of $(T(\mathfrak{a}), m_{T(\mathfrak{a})}, \Delta^0_{T(\mathfrak{a})}, \delta_{T(\mathfrak{a})}, \varepsilon_{T(\mathfrak{a})}, S_{T(\mathfrak{a})})$.*

Then there exists an algebra isomorphism $\theta : T \to T(\mathfrak{a})[[\hbar]]$, such that if $\Delta : T(\mathfrak{a})[[\hbar]] \to T(\mathfrak{a})^{\otimes 2}[[\hbar]]$ is the map $(\theta \otimes \theta) \circ \Delta_T \circ \theta^{-1}$, then $\Delta(T^i(\mathfrak{a})) \subset \widehat{\bigoplus}_{p,q|p+q \geq i} \hbar^{p+q-i} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a})$ ($\widehat{\oplus}$ is the completed direct sum).

Moreover, any isomorphism with this property is of the form $\kappa \circ \theta$, where κ is the automorphism of $T(\mathfrak{a})[[\hbar]]$ induced by a map $\mathfrak{a} \to \widehat{\bigoplus}_{n \geq 1} \hbar^{n-1} T^n(\mathfrak{a})[[\hbar]]$ whose reduction modulo $\hbar$ is the identity.

Proof. Let $\phi : \mathfrak{a} \to T$ be any section of the projection map $T \to T/\hbar T = T(\mathfrak{a}) \to \mathfrak{a}$ (the second map is the projection on $\mathfrak{a}$ parallel to $\oplus_{i \neq 1} T^i(\mathfrak{a})$). This map extends to a unique algebra morphism $\iota : T(\mathfrak{a})[[\hbar]] \to T$. Since T is a flat deformation of $T(\mathfrak{a})$, ι is a linear isomorphism. Let us set $\Delta^\iota_T = (\iota \otimes \iota) \circ \Delta_T \circ \iota^{-1}$. Then Δ^ι_T is a coproduct map on $T(\mathfrak{a})[[\hbar]]$. We will now construct an algebra automorphism ϕ of $T(\mathfrak{a})[[\hbar]]$, such that the reduction of $\phi \mod \hbar$ is the identity and $(\phi \otimes \phi) \circ \Delta^\iota_T \circ \phi^{-1}$ has the required property.

Assume that we constructed an automorphism $\phi_{(n)}$ of $T(\mathfrak{a})[[\hbar]]$, whose reduction modulo $\hbar$ is the identity, and such that $\Delta_{T,n} = (\phi_{(n)} \otimes \phi_{(n)}) \circ \Delta^\iota_T \circ \phi^{-1}_{(n)}$ has the property

$$\Delta_{T,n}(\mathfrak{a}) \subset \bigoplus_{p,q|p+q \geq 1} \hbar^{p+q-1} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a}) + \hbar^n T(\mathfrak{a}) \otimes T(\mathfrak{a})[[\hbar]]. \qquad (4)$$

Let $\Delta_{T,n,k}$ be the linear endomorphisms of $T(\mathfrak{a})$ such that $\Delta_{T,n} = \sum_{k \geq 0} \hbar^k \Delta_{T,n,k}$. If we denote by $\Delta_{T(\mathfrak{a})}$ the usual (undeformed) coproduct of $T(\mathfrak{a})$, the coassociativity of $\Delta_{T,n}$ implies that $(\Delta_{T(\mathfrak{a})} \otimes \text{id} - \text{id} \otimes \Delta_{T(\mathfrak{a})}) \circ \Delta_{T,n,n}$ maps $T(\mathfrak{a})$ to $\oplus_{p,q,r|p+q+r \leq n+1} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a}) \otimes T^r(\mathfrak{a})$.

Let us denote by $\Delta_{>n+1}$ the composition of $(\Delta_{T,n,n})_{|\mathfrak{a}}$ with the projection of $T(\mathfrak{a})^{\otimes 2}$ on $\oplus_{p,q|p+q>n+1} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a})$ parallel to $\oplus_{p,q|p+q \leq n+1} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a})$. Then $(\Delta_{T(\mathfrak{a})} \otimes \text{id} - \text{id} \otimes \Delta_{T(\mathfrak{a})}) \circ \Delta_{>n+1} = 0$. Let ψ_n be the composition $-\varepsilon_{T(\mathfrak{a})} \circ \Delta_{>n+1}$; ϕ'_n is a linear map from $\mathfrak{a}$ to $T(\mathfrak{a})$. Then the relation

$$\text{id}_{T(\mathfrak{a}) \otimes T(\mathfrak{a})} = \Delta_{T(\mathfrak{a})} \circ (\varepsilon_{T(\mathfrak{a})}) + (\varepsilon \otimes \text{id}^{\otimes 2}_{T(\mathfrak{a})}) \circ (\Delta_{(T(\mathfrak{a}))} \otimes \text{id}_{T(\mathfrak{a})} - \text{id}_{T(\mathfrak{a})} \otimes \Delta_{(T(\mathfrak{a}))})$$

implies that $\Delta_{>n+1} = \Delta_{T(\mathfrak{a})} \circ \psi_n$. Let us define ϕ'_n as the automorphism of $T(\mathfrak{a})[[\hbar]]$ induced by the map $1 - \hbar^n \psi_n$, and set $\phi_{(n+1)} = \phi'_n \circ \phi_{(n)}$, then $(\phi_{(n+1)} \otimes \phi_{(n+1)}) \circ \Delta_{T,n} \circ (\phi_{(n+1)})^{-1}$ satisfies (4) with n replaced by $n+1$.

The sequence $(\phi_{(n)})_{n \geq 0}$ has a unique $\hbar$-adic limit, wich we denote by ϕ. Then if we set $\Delta_{T,\infty} = (\phi_{(\infty)} \otimes \phi_{(\infty)}) \circ \Delta_T^\iota \circ \phi_{(\infty)}^{-1}$, then we have

$$\Delta_{T,\infty}(\mathfrak{a}) \subset \bigoplus_{p,q|p+q \geq 1} \hbar^{p+q-1} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a}).$$

Since $\Delta_{T,\infty}$ is an algebra morphism, this relation implies

$$\Delta_{T,\infty}(T^k(\mathfrak{a})) \subset \bigoplus_{p,q|p+q \geq k} \hbar^{p+q-k} T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a})$$

for any k. So we set $\theta = \phi \circ \iota$. $\square$

Assume that $(\mathfrak{a}, \delta_{\mathfrak{a}})$ is a finite-dimensional (or nonnegatively graded with finite dimensional components) Lie coalgebra. Then $\mathfrak{a}^*$ (or the restricted dual of $\mathfrak{a}$) is a Lie algebra; denote it by $\mathfrak{b}$. In the situation of Proposition 1.2, define $\delta_{pq}^{\mathfrak{b}}$ as the composition of the restriction $(\theta \otimes \theta) \circ \Delta \circ \theta^{-1}$ with the projection of $T(\mathfrak{a}) \otimes T(\mathfrak{a})[[\hbar]]$ on $T^p(\mathfrak{a}) \otimes T^q(\mathfrak{a})[[\hbar]]$ parallel to all other $T^{p'}(\mathfrak{a}) \otimes T^{q'}(\mathfrak{a})[[\hbar]]$.

Define $B_{pq}^{\mathfrak{b}}$ as the linear map from $\mathfrak{b}^{\otimes p+q}$ to $\mathfrak{b}[[\hbar]]$ dual to $\delta_{pq}^{\mathfrak{b}}$. Equip $\mathrm{Sh}(\mathfrak{b})[[\hbar]]$ with its usual coproduct and the product $m_{T,\theta}$ given by formula (3), with the B_{pq} replaced by $B_{pq}^{\mathfrak{b}}$.

Lemma 1.1. *In the situation of Proposition 1.2, there is a unique Hopf algebra structure* $(\mathrm{Sh}(\mathfrak{b})[[\hbar]], m_{T,\theta}, \Delta_{\mathrm{Sh}(\mathfrak{b})}, \varepsilon_{\mathrm{Sh}(\mathfrak{b})}, S_{T,\theta})$ *on* $\mathrm{Sh}(\mathfrak{b})[[\hbar]]$ *with product* $m_{T,\theta}$ *and coproduct* $\Delta_{\mathrm{Sh}(\mathfrak{b})}$. *The pairing* $T(\mathfrak{a})[[\hbar]] \times \mathrm{Sh}(\mathfrak{b})[[\hbar]] \to \mathbb{K}[[\hbar]][\hbar^{-1}]$ *defined by* $\langle a_1 \otimes \cdots \otimes a_p, (b_1 \ldots b_q) \rangle = \frac{\delta_{pq}}{\hbar^p} \prod_{i=1}^p \langle a_i, b_i \rangle$ *induces a Hopf pairing between this Hopf algebra and the Hopf algebra structure on* $T(\mathfrak{a})[[\hbar]]$ *defined in Proposition 1.2.*

We now construct a map $\gamma_{\mathbb{K}} : \mathrm{III}(\mathbb{K}) \to \{$universal quantization functors of the tensor algebras $T(\mathfrak{a})\}$. If n is an integer and P belongs to $FL_n(\mathbb{K})$, and if $(\mathfrak{a}, \delta_{\mathfrak{a}})$ is a Lie coalgebra, let us define the map $\delta_{\mathfrak{a}}^{(P)} : \mathfrak{a} \to \mathfrak{a}^{\otimes n}$ as follows. Let $FA_n(\mathbb{K})$ be the part of the free algebra with n generators, homogeneous in each generator. There is a unique element $(P_\sigma)_{\sigma \in \mathfrak{S}_n}$ in $\mathbb{K}^{\mathfrak{S}_n}$ such that $P = \sum_{\sigma_i n \mathfrak{S}_n} P_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$. Let us set

$$\delta_{\mathfrak{a}}^{(P)}(a) = \frac{1}{n} \left((\mathrm{id}_{\mathfrak{a}}^{\otimes n} \otimes \delta_{\mathfrak{a}}) \circ \cdots \circ (\mathrm{id}_{\mathfrak{a}} \otimes \delta) \circ \delta(a) \right)^{(\sigma(1)\ldots\sigma(n))},$$

for any a in $\mathfrak{a}$. Then Proposition D.1 implies that if $\mathfrak{b}$ is any Lie algebra dual to $\mathfrak{a}$, then

$$\langle \delta_{\mathfrak{a}}^{(P)}(a), b_1 \otimes \cdots \otimes b_n \rangle_{\mathfrak{a}^{\otimes n} \times \mathfrak{b}^{\otimes n}} = \langle a, P(b_1, \ldots, b_n) \rangle_{\mathfrak{a}^{\otimes n} \times \mathfrak{b}^{\otimes n}}$$

(here $\langle \ , \ \rangle_{\mathfrak{a} \times \mathfrak{b}}$ is the pairing between $\mathfrak{a}$ and $\mathfrak{b}$ and $\langle \ , \ \rangle_{\mathfrak{a}^{\otimes n} \times \mathfrak{b}^{\otimes n}}$ is its nth tensor power). If P belongs to $\mathbb{K}[[\hbar]]$, the map $\delta^{(P)} : \mathfrak{a} \to \mathfrak{a}^{\otimes n}[[\hbar]]$ is defined in the same way.

If ω belongs to $\mathcal{B}(\mathbb{K}[[\hbar]])$, and $\mathfrak{a}$ is any Lie coalgebra, let us define $T_\hbar^\omega(\mathfrak{a})$ as follows. As an algebra, $T_\hbar^\omega(\mathfrak{a})$ is isomorphic to $T(\mathfrak{a})[[\hbar]]$. Define μ_{pq} as the map from $\mathfrak{a}^{\otimes p+q}$ to $T(\mathfrak{a}) \otimes T(\mathfrak{a})$ obtained by the composition of the isomorphism $\mathfrak{a}^{\otimes(p+q)} \to \mathfrak{a}^{\otimes p} \otimes \mathfrak{a}^{\otimes q}$ with the injections of $\mathfrak{a}^{\otimes p}$ and $\mathfrak{a}^{\otimes q}$ in each factor of $T(\mathfrak{a}) \otimes T(\mathfrak{a})$. Then there exists a unique coalgebra map $\Delta_{T_\hbar^\omega(\mathfrak{a})}$ on $T(\mathfrak{a})[[\hbar]]$, such that for any a in $\mathfrak{a}$,

$$\Delta_{T_\hbar^\omega(\mathfrak{a})}(a) = \sum_{p,q \geq 0} \hbar^{p+q-1} \mu_{p,q}(\delta^{(B_{pq})}(a)).$$

The usual augmentation map $\varepsilon_{T_\hbar^\omega(\mathfrak{a})}$ is a counit for this coalgebra structure, and there exists a unique corresponding antipode $S_{T_\hbar^\omega(\mathfrak{a})}$. $(T_\hbar^\omega(\mathfrak{a}), m_{T_\hbar^\omega(\mathfrak{a})}, \Delta_{T_\hbar^\omega(\mathfrak{a})}, \varepsilon_{T_\hbar^\omega(\mathfrak{a})}, S_{T_\hbar^\omega(\mathfrak{a})})$ is then a quantized tensor algebra. It is easy to see that it is a quantization of $(T(\mathfrak{a}), m_{T(\mathfrak{a})} \Delta_{T(\mathfrak{a})}, \delta_{T(\mathfrak{a})})$. Moreover, for any fixed ω, the map $\mathfrak{a} \mapsto T_\hbar^\omega(\mathfrak{a})$ is a quantization functor of the tensor algebras $T(\mathfrak{a})$, and if ω and ω' are in the same $\mathcal{G}(\mathbb{K}[[\hbar]])$-orbit, then there are functorial (in $\mathfrak{a}$) Hopf algebra isomorphisms between $T_\hbar^\omega(\mathfrak{a})$ and $T_\hbar^{\omega'}(\mathfrak{a})$. If ϖ belongs to $\text{ш}(\mathbb{K})$, let us define $(\mathfrak{a} \mapsto T_\hbar^\varpi(\mathfrak{a}))$ as any of the quantization functors $(\mathfrak{a} \mapsto T_\hbar^\omega(\mathfrak{a}))$, where ϖ is the coset of ω. Summarizing, we have

Proposition 1.3. *The assignment $\varpi \mapsto (\mathfrak{a} \mapsto T_\hbar^\varpi(\mathfrak{a}))$ defines a map $\gamma_\mathbb{K}$ from $\text{ш}(\mathbb{K})$ to {universal quantization functors of the tensor algebras $T(\mathfrak{a})$}.*

We will now show

Proposition 1.4. *The restriction to $\text{ш}(\mathbb{K})_{1/2}$ of the map $\gamma_\mathbb{K}$ is a bijection. It is functorial in $\mathbb{K}$ and equivariant with respect to the natural actions of $\text{Aut}(\mathbb{K}[[\hbar]])$.*

Proof. Let us construct the inverse map to $\gamma_\mathbb{K}$.

Let $\mathcal{F}_n$ be the free Lie algebra with n generators $x_1, \ldots, x_n$. If Ψ is a quantization functor of the tensor algebras $T(\mathfrak{a})$, we may apply Proposition 1.2 and Lemma 1.1 to $\Psi(\mathcal{F}_n^*)$. These statements have analogues in the situation of $\mathbb{N}^n$-graded Lie algebras. Functoriality with respect to the Lie algebra automorphisms of $\mathcal{F}_n$ given by $x_i \mapsto \lambda_i x_i$ allows us to apply these refined statements to $\mathcal{F}_n$. We obtain maps $B_{pq}^{\mathcal{F}_n} : (\mathcal{F}_n)^{\otimes p+q} \to \mathcal{F}_n[[\hbar]]$, graded and well defined up to graded automorphisms. The maps $B_{pq}^{\mathcal{F}_n}$ are graded, so if $p+q \leq n$, then $B_{pq}^{\mathcal{F}_n}(x_1, \ldots, x_{p+q})$ belongs to $FL_{p+q}(\mathbb{K}[[\hbar]])$. The associativity of the product of $\Psi(\mathcal{F}_n)$ implies that the $(B_{pq}^{\mathcal{F}_n})_{p,q|p+q \leq n}$ satisfy identities (1), for $p+q+r \leq n$. The family $(B_{pq}^{\mathcal{F}_n})_{p,q|p+q \leq n}$ therefore defines an element $\widetilde{\text{ш}}_n(\Psi)$ of $\text{ш}_n(\mathbb{K})$. Functoriality with respect to the map $\mathcal{F}_n \to \mathcal{F}_{n+1}$ sending each x_i to x_i implies that the family $(\widetilde{\text{ш}}_n(\Psi))_{n \geq 0}$ defines an element $\widetilde{\text{ш}}(\Psi)$ of $\text{ш}_n(\mathbb{K})$. Then there is a unique $\lambda(\Psi) \in 1 + o(\hbar)$ such that $\widetilde{\text{ш}}(\Psi)$ belongs to $\text{ш}_n(\mathbb{K})_{\lambda(\Psi)/2}$. Let $\text{ш}(\Psi)$ be the result of the action of $\lambda(\Psi)^{-1} \in \mathbb{K}[[\hbar]]^\times$ on $\widetilde{\text{ш}}(\Psi)$. We have obviously $\text{ш} \circ \gamma_\mathbb{K} = \text{id}_{\text{ш}(\mathbb{K})_{1/2}}$.

Let us prove that $\gamma_\mathbb{K} \circ \text{ш} = \text{id}_{\text{ш}(\mathbb{K})_{1/2}}$. Let Ψ be a universal quantization functor of the tensor algebras $T(\mathfrak{a})$. Apply Proposition 1.2 and Lemma 1.1 to each Lie coalgebra $\mathfrak{a}$. Then for each Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, we obtain linear maps

$B_{pq}^{\mathfrak{g}} : \mathfrak{g}^{\otimes p+q} \to \mathfrak{g}[[\hbar]]$ satisfying the relations (1), depending polynomially in the structure constants of $\mathfrak{g}$, and such that the dual to $\Psi(\mathfrak{a})$ is the quantized shuffle algebra of $\mathfrak{a}^*$ corresponding to the maps $B_{pq}^{\mathfrak{g}}$. One the other hand, such linear maps $\mathfrak{g}^{\otimes p+q} \to \mathfrak{g}$ all correspond to substitution of elements of $\mathfrak{g}$ in free Lie polynomials. The family $(B_{pq}^{\mathfrak{g}})_{p,q \geq 0}$ therefore corresponds to an element of $\text{Ш}(\mathbb{K})$. □

1.5. **Relation with the CBH formula.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and let Sym : $S\mathfrak{g} \to U\mathfrak{g}$ be the symmetrization map, such that for any x in $\mathfrak{g}$ and any integer n, $\text{Sym}(x^n) = x^n$. The pull-back of the product on the universal enveloping algebra $U\mathfrak{g}$ by the symmetrization map Sym may be viewed as a star-product over $S\mathfrak{g}$. This pull-back involves the Campbell-Baker-Hausdorff (CBH) series and is called PBW quantization of $S\mathfrak{g}$.

Let us recall how the CBH series is defined. It is an infinite series $B^{CBH}(x,y) = \sum_{p,q \geq 0} B_{pq}^{CBH}(x,y)$, where $B_{pq}^{CBH}(x,y)$ is an element of the free Lie algebra in two variables x, y, homogeneous of degrees p in x and q in y. For x, y formal variables in the neighborhood of zero in $\mathfrak{g}$, the identity

$$e^x e^y = e^{B^{CBH}(x,y)}$$

holds in the formal Lie group associated with $\mathfrak{g}$.

Then if x and y belong to $\mathfrak{g}$, we have the identity

$$x^p y^q = \sum_{k \geq 0} \frac{p! q!}{k!} \sum_{(p_1,\dots,p_k) \in \text{Part}_k(p), (q_1,\dots,q_k) \in \text{Part}_k(q)} \text{Sym}(B_{p_1 q_1}^{CBH}(x,y) \cdots B_{p_k q_k}^{CBH}(x,y)) \tag{5}$$

in $U\mathfrak{g}$.

Assume that $\omega = (B_{pq})_{p,q \geq 0}$ belongs to $\mathcal{B}(\mathbb{K}[[\hbar]])_{1/2}$. Since the image of x^p by the canonical map $U\mathfrak{g} \to \text{Sh}_{\hbar}^{\omega}(\mathfrak{g})$ is $p!(x \dots x)$ (x appears p times), and identifying the terms of lower degree of the image of (5) by $U\mathfrak{g} \to \text{Sh}_{\hbar}^{\omega}(\mathfrak{g})$, we get $B_{pq}(x, \dots, x|y, \dots, y) = B^{CBH}(x,y)$. This identity holds for any Lie algebra, in particular, if $\mathfrak{g}$ is the free Lie algebra with two generators and x, y are identified with these generators. We have therefore

Proposition 1.5. *If $\omega = (B_{pq})_{p,q \geq 0}$ belongs to $\mathcal{B}(\mathbb{K}[[\hbar]])_{1/2}$, then the identities*

$$B_{pq}(x, \dots, x|y, \dots, y) = B_{pq}^{CBH}(x,y)$$

hold in the free Lie algebra with two generators x, y;

1.6. **Explicit formulas.** One may show that if ω is any element in $\mathcal{B}(\mathbb{K}[[\hbar]])_{1/2}$, there exists $\omega' = (B_{pq})_{p,q \geq 0}$ in the orbit $\mathcal{G}(\mathbb{K}[[\hbar]]) * \omega$ of ω such that

$$B_{12}(x, x'|y) = \frac{1}{24}([x,[x',y]] + [x',[x,y]]), \quad B_{21}(x|y,y') = \frac{1}{24}([y,[y',x]] + [y',[y,x]]).$$

In the algebra $\text{Sh}^{\omega'}(\mathfrak{g})$, we have the relations

$$(x)(y) = (xy) + (yx) + \frac{1}{2}([x,y]),$$

$$(xx')(y) = (xx'y) + (xyx') + (yxx') + \frac{1}{2}(x[x', y]) + \frac{1}{2}([x, y]x') + \frac{1}{24}([x, [x', y]] + [x', [x, y]]),$$

$$(x)(yy') = (xyy') + (yxy') + (yy'x) + \frac{1}{2}([x, y]y') + \frac{1}{2}(y[x, y']) + \frac{1}{24}([y, [y', x]] + [y', [y, x]]).$$

2. R-MATRICES IN SHUFFLE ALGEBRAS

Let us fix an element ω of $\mathcal{B}(\mathbb{K}[[\hbar]])$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras and let $r_{\mathfrak{ab}} = \sum_{i \in I} a_i \otimes b_i$ be an element of $\mathfrak{a} \otimes \mathfrak{b}$. In this Section, we associate to $r_{\mathfrak{ab}}$ a family of elements $(\mathcal{R}_n^\omega(r_{\mathfrak{ab}}))_{n \geq 0}$ of $\mathrm{Sh}^\omega(\mathfrak{a}) \otimes \mathrm{Sh}^\omega(\mathfrak{b})$, satisfying analogues of the quasitriangular identities $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{(13)}\mathcal{R}^{(23)}$ and $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{(13)}\mathcal{R}^{(12)}$.

If $n_1, \dots, n_k$ are positive integers, denote by $x_1^{(i)}, \dots, x_{n_i}^{(i)}$ the generators of the ith factor of the tensor product $(\otimes_{i=1}^k FL_{n_i}) \otimes FL_{n_1+\cdots+n_k}$ and by $y_1, \dots, y_{n_1+\cdots+n_k}$ the generators of the last tensor factor. Then the product of symmetric groups $\prod_{i=1}^k \mathfrak{S}_{n_i}$ acts on this tensor product, is such a way that $\mathfrak{S}_i$ permutes simultaneously the generators $(x_a^{(i)})_{a=1,\dots,k_i}$ and $(y_{n_1+\cdots+n_{i-1}+a})_{a=1,\dots,k_i}$. Let us denote by $F_{n_1,\dots,n_k}$ the space of coinvariants of this action, so

$$F_{n_1,\dots,n_k} = \left((\bigotimes_{i=1}^k FL_{n_i}) \otimes FL_{n_1+\cdots+n_k} \right)_{\prod_{i=1}^k \mathfrak{S}_{n_i}}.$$

Let us define inductively $\lambda_{n_1,\dots,n_k} \in F_{n_1,\dots,n_k}$ as follows. Recall that if p and r are integers, $\mathrm{Part}_r(p)$ is the set of r-partitions of p, i.e. the set of families of integers $(p_1, \dots, p_r)$ such that $p_1 + \cdots + p_r = p$. We set $\lambda_1 = x_1^{(1)} \otimes y_1$, and if $\sum_{l=1}^k n_l > 1$

$$\lambda_{n_1,\dots,n_k} = \frac{1}{n_1 + \cdots + n_k - 1} \sum_{\alpha_1,\dots,\alpha_k \in \mathbb{N}_{>0}} \sum_{l=1}^{k-1}$$

$$\sum_{(n_{ij})_j \in \mathrm{Part}_{\alpha_i}(n_i)} \lambda''_{(n_{1i})_i} \otimes \cdots \otimes \lambda''_{(n_{ki})_i} \otimes B_{\alpha_1+\cdots+\alpha_l, \alpha_{l+1}+\cdots+\alpha_k}^{\omega^\vee} \left(\lambda'^{(1)}_{(n_{1j})_j}, \dots, \lambda'^{(\alpha_k)}_{(n_{kj})_j} \right),$$

where we set

$$\lambda_{(n_{ij})_{j=1,\dots,\alpha_i}} = \lambda'^{(1)}_{(n_{ij})_{j=1,\dots,\alpha_i}} \otimes \cdots \otimes \lambda'^{(\alpha_i)}_{(n_{ij})_{j=1,\dots,\alpha_i}} \otimes \lambda''_{(n_{ij})_{j=1,\dots,\alpha_i}},$$

and the variables $(x_a^{(i)})_{a=1,\dots,n_i}$ (resp., $y_{\sum_{\alpha<i} n_\alpha + \sum_{\beta<j} n_{i\beta}+1}, \cdots, y_{\sum_{\alpha<i} n_\alpha + \sum_{\beta \leq j} n_{i\beta}}$) are substituted in $\lambda''_{(n_{ij})_j}$ (resp., in $\lambda'^{(j)}_{(n_{ij})_j}$).

Let us denote by $\mathrm{conc}_{\mathrm{Sh}^\omega(\mathfrak{b})}$ the concatenation product on the tensor algebra $T(\mathfrak{b})[[\hbar]]$, which we identify linearly with $\mathrm{Sh}^\omega(\mathfrak{b})$. We have $\mathrm{conc}_{\mathrm{Sh}^\omega(\mathfrak{b})}((x_1 \dots x_k) \otimes (x_{k+1} \dots x_n)) = (x_1 \dots x_n)$.

There is a unique map $\kappa(r_{\mathfrak{ab}})$ from $F_{n_1,\dots,n_k}$ to $\mathrm{Sh}^\omega(\mathfrak{a}) \otimes \mathfrak{b}$, such that for any $P_1 \in FL_{n_1}, \dots, P_k \in FL_{n_k}$, $Q \in FL_{n_1+\cdots+n_k}$, we have

$$\kappa(r_{\mathfrak{ab}})(\otimes_{i=1}^k P_i(x_a^{(i)}) \otimes Q(y_1, \dots, y_{n_1+\cdots+n_a})) = \sum_{i_1,\dots i_{n_1+\cdots+n_k} \in I}$$

$$\left(P_1(a_{i_1}, \dots, a_{i_{n_1}}) \dots P_k(a_{i_{n_1+\cdots+n_{k-1}+1}}, \dots, a_{i_{n_1+\cdots+n_k}}) \right) \otimes Q(b_{i_1}, \dots, b_{i_{n_1+\cdots+n_k}}).$$

We also denote by $\kappa(r_{\mathfrak{ab}})$ the unique extension of this map to a linear map from $\oplus_{n_1,\dots,n_k} F_{n_1,\dots,n_k}$ to $\mathrm{Sh}^\omega(\mathfrak{a}) \otimes \mathfrak{b}$.

Let us denote by $\Psi_{\mathrm{Sh}(\mathfrak{b})}$ the linear automorphism of $\mathrm{Sh}(\mathfrak{b})$ such that for any $x_1, \dots, x_n$ in $\mathfrak{b}$, $\Psi_{\mathrm{Sh}(\mathfrak{b})}((x_1 \dots x_n)) = (x_n \dots x_1)$.

Lemma 2.1. *If $\Delta'_{\mathrm{Sh}(\mathfrak{b})}$ is the composition of $\Delta_{\mathrm{Sh}(\mathfrak{b})}$ with the permutation of factors, then $\Psi_{\mathrm{Sh}(\mathfrak{b})}$ is a Hopf algebra isomorphism from $(\mathrm{Sh}^{\omega^\vee}(\mathfrak{b}), m_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})}, \Delta_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})})$ to $(\mathrm{Sh}^{\omega}(\mathfrak{b}), m_{\mathrm{Sh}^{\omega}(\mathfrak{b})}, \Delta'_{\mathrm{Sh}^{\omega}(\mathfrak{b})})$, so $m_{\mathrm{Sh}^{\omega}(\mathfrak{b})} \circ (\Psi_{\mathrm{Sh}(\mathfrak{b})} \otimes \Psi_{\mathrm{Sh}(\mathfrak{b})}) = \Psi_{\mathrm{Sh}(\mathfrak{b})} \circ m_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})}$ and $\Delta'_{\mathrm{Sh}^{\omega}(\mathfrak{b})} \circ \Psi_{\mathrm{Sh}(\mathfrak{b})} = (\Psi_{\mathrm{Sh}(\mathfrak{b})} \otimes \Psi_{\mathrm{Sh}(\mathfrak{b})}) \circ \Delta_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})}$.*

If m is an associative operation, we will denote by $m^{(k)}$ the kth fold product associated to m. If n is a positive integer, we set $\lambda_n = \sum_{k>0} \sum_{(n_1, \dots, n_k) \in \mathrm{Part}_k(n)} \lambda_{n_1, \dots, n_k}$.

Proposition 2.1. *Let us define the family $(\mathcal{R}'_n(r_{\mathfrak{ab}}))_{n \geq 0}$ of elements of $\mathrm{Sh}^{\omega}(\mathfrak{a}) \otimes \mathrm{Sh}^{\omega^\vee}(\mathfrak{b})$ as follows. $\mathcal{R}'_0(r_{\mathfrak{ab}}) = 1$, $\mathcal{R}'_1(r_{\mathfrak{ab}}) = r_{\mathfrak{ab}}$ and*

$$\mathcal{R}'_n(r_{\mathfrak{ab}})$$
$$= \sum_{k \geq 1} \sum_{(n_1, \dots, n_k) \in \mathrm{Part}_k(n)} \left(m^{(k)}_{\mathrm{Sh}(\mathfrak{a})} \otimes \mathrm{conc}^{(k)}_{\mathrm{Sh}(\mathfrak{b})} \right) \left(\kappa(r_{\mathfrak{ab}})(\lambda_{n_1})^{(1, k+1)} \cdots \kappa(r_{\mathfrak{ab}})(\lambda_{n_k})^{(k, 2k)} \right).$$

If we set

$$\mathcal{R}^{\omega}_n(r_{\mathfrak{ab}}) = (\mathrm{id} \otimes \Psi_{\mathrm{Sh}(\mathfrak{b})})(\mathcal{R}'_n(r_{\mathfrak{ab}})),$$

then the family $(\mathcal{R}_n(r_{\mathfrak{ab}}))_{n \geq 0}$ satisfies

$$(\Delta_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id})(\mathcal{R}^{\omega}_n(r_{\mathfrak{ab}})) = \sum_{k=0}^{n} (\mathcal{R}^{\omega}_k(r_{\mathfrak{ab}}))^{(13)} (\mathcal{R}^{\omega}_{n-k}(r_{\mathfrak{ab}}))^{(23)}, \qquad (6)$$

$$(\mathrm{id} \otimes \Delta_{\mathrm{Sh}^{\omega}(\mathfrak{b})})(\mathcal{R}_n(r_{\mathfrak{ab}})) = \sum_{k=0}^{n} (\mathcal{R}^{\omega}_k(r_{\mathfrak{ab}}))^{(13)} (\mathcal{R}^{\omega}_{n-k})(r_{\mathfrak{ab}})^{(12)}.$$

Moreover, we have

$$(S_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}^{\omega}(\mathfrak{b})})(\mathcal{R}^{\omega}_n(r_{\mathfrak{ab}})) = (\mathrm{id}_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes S^{-1}_{\mathrm{Sh}^{\omega}(\mathfrak{b})})(\mathcal{R}^{\omega}_n(r_{\mathfrak{ab}})). \qquad (7)$$

Proof. It is immediate that $(\mathcal{R}'_n(r_{\mathfrak{ab}}))_{n \geq 0}$ satisfies

$$(\mathrm{id} \otimes \Delta_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})})(\mathcal{R}'_n(r_{\mathfrak{ab}})) = \sum_{k=0}^{n} \mathcal{R}_k(r_{\mathfrak{ab}})'^{(12)} \mathcal{R}_{n-k}(r_{\mathfrak{ab}})'^{(13)}, \qquad (8)$$

so $(\mathcal{R}'_n(r_{\mathfrak{ab}}))_{n \geq 0}$ satisfies the second part of (6). Let us prove by induction on m that the relation

$$(\Delta_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id})(\mathcal{R}'_m(r_{\mathfrak{ab}})) = \sum_{k=0}^{m} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{m-k}(r_{\mathfrak{ab}})^{(23)} \qquad (9)$$

is also satisfied. The relation clearly holds for $m = 0, 1$. Assume that it is satisfied up to order $m = n - 1$. Let us set

$$Z = (\Delta_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id})(\mathcal{R}'_n(r_{\mathfrak{ab}})) - \sum_{k=0}^{n} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{n-k}(r_{\mathfrak{ab}})^{(23)}$$

and let us apply $(\mathrm{id} \otimes \Delta_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})})$ to Z. We find

$$(\mathrm{id} \otimes \Delta_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})})(Z) = (\Delta_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \Delta_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})})(\mathcal{R}'_n(r_{\mathfrak{ab}}))$$

$$- \sum_{k=0}^n \sum_{k',k''} \mathcal{R}'_{k'}(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{k-k'}(r_{\mathfrak{ab}})^{(14)} \mathcal{R}'_{k''}(r_{\mathfrak{ab}})^{(23)} \mathcal{R}'_{n-k-k''}(r_{\mathfrak{ab}})^{(24)},$$

and equation (8) and the induction hypothesis imply that this is $Z^{(123)} + Z^{(124)}$. Since the space of primitive elements of $\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})$ coincides with $\mathfrak{b}$, Z belongs to $\mathrm{Sh}^\omega(\mathfrak{a})^{\otimes 2} \otimes \mathfrak{b}$.

Let us show that Z is zero. Let us set $\widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})}(x) = \Delta_{\mathrm{Sh}^\omega(\mathfrak{a})}(x) - x \otimes 1 - 1 \otimes x + \varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})}(x)(1 \otimes 1)$. The restriction of $\widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})}$ to $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})$ is a linear map from $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})$ to $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})^{\otimes 2}$. Let us define $\widetilde{\mathrm{conc}}_{\mathrm{Sh}_{\mathfrak{a}}}$ as the linear map from $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})^{\otimes 2}$ to $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})$ such that if $y = \sum_{k \geq 2} y_k$, with $y_k \in \oplus_{k',k''|k'+k''=k} \mathfrak{a}^{\otimes k'} \otimes \mathfrak{a}^{\otimes k''}$, then $\widetilde{\mathrm{conc}}_{\mathrm{Sh}^\omega(\mathfrak{a})}(y) = \sum_{k \geq 2} \frac{1}{k-1} \mathrm{conc}(y_k)$. Let us also denote by $\widetilde{\mathrm{conc}}^{(2)}$ the map from $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})^{\otimes 3}$ to $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})^{\otimes 2}$ whose restriction to $\oplus_{k,k',k''|k+k'+k''=n} \mathfrak{a}^{\otimes k} \otimes \mathfrak{a}^{\otimes k'} \otimes \mathfrak{a}^{\otimes k''}$ is equal to $\frac{1}{n-1}(\mathrm{conc}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{conc}_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})})$. Then we have the following homotopy formula

$$\mathrm{id}_{\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})^{\otimes 2}} = \widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})} \circ \widetilde{\mathrm{conc}}_{\mathrm{Sh}(\mathfrak{b})} + \widetilde{\mathrm{conc}}^{(2)} \circ (\widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id} - \mathrm{id} \otimes \widetilde{\Delta}_{\mathrm{Sh}(\mathfrak{a})}). \tag{10}$$

$(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id} \otimes \mathrm{id})(Z) = (\mathrm{id} \otimes \varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id})(Z) = 0$, so Z belongs to $\mathrm{Ker}(\varepsilon_{\mathrm{Sh}^\omega(\mathfrak{a})})^{\otimes 2} \otimes \mathfrak{b}$. Let us apply identity (10) to the first two tensor factors of Z. Since $(\widetilde{\mathrm{conc}}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id}_{\mathfrak{b}})(Z)$ belongs to $\mathrm{Sh}^\omega(\mathfrak{a})^{\otimes 2} \otimes \mathfrak{b}$, this term coincides with $(\widetilde{\mathrm{conc}}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{pr}_{\mathfrak{b}})(Z)$, therefore it is equal to the sum of $(n-1)\kappa(r_{\mathfrak{ab}})(\lambda_n)$ and contributions of the $\mathcal{R}'_k(r_{\mathfrak{ab}})$, where $k < n$. By the construction of λ_n, this sum is zero.

Let us compute now

$$\left((\widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{a})} - \mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})}) \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})}\right)(Z). \tag{11}$$

Recall that Z is equal to $(\widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})})(\mathcal{R}'_n(r_{\mathfrak{ab}})) - \sum_{k=1}^{n-1} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{n-k}(r_{\mathfrak{ab}})^{(23)}$; since $(\widetilde{\Delta}_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})} - \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})} \otimes \widetilde{\Delta}_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})}) \circ \widetilde{\Delta}_{\mathrm{Sh}^{\omega^\vee}(\mathfrak{b})} = 0$, (11) is equal to

$$-\left((\widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})} - \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})} \otimes \widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})}) \otimes \mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{b})}\right)\left(\sum_{k=1}^{n-1} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{n-k}(r_{\mathfrak{ab}})^{(23)}\right).$$

The induction hypothesis implies that

$$\left(\widetilde{\Delta}_{\mathrm{Sh}(^\omega\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})}\right)\left(\sum_{k=1}^{n-1} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{n-k}(r_{\mathfrak{ab}})^{(23)}\right)$$

and $\left(\mathrm{id}_{\mathrm{Sh}(\mathfrak{a})} \otimes \widetilde{\Delta}_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})}\right)\left(\sum_{k=1}^{n-1} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(13)} \mathcal{R}'_{n-k}(r_{\mathfrak{ab}})^{(23)}\right)$ are both equal to $\sum_{k,k',k''|k,k',k''>0, k+k'+k''=n} \mathcal{R}'_k(r_{\mathfrak{ab}})^{(14)} \mathcal{R}'_{k'}(r_{\mathfrak{ab}})^{(24)} \mathcal{R}'_{k''}(r_{\mathfrak{ab}})^{(34)}$. Therefore (11) vanishes. The homotopy formula (10) then implies that Z is zero. This proves the induction step.

Equations (6) imply that if t is a formal parameter and $\mathcal{R}^\omega(r_{\mathfrak{ab}}) = \sum_{n \geq 0} t^n \mathcal{R}_n^\omega(r_{\mathfrak{ab}})$, we have $(S_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id})(\mathcal{R}^\omega(r_{\mathfrak{ab}})) = \mathcal{R}^\omega(r_{\mathfrak{ab}})^{-1}$; as $S^{-1}_{\mathrm{Sh}^\omega(\mathfrak{b})}$ is the antipode corresponding to the coproduct $\Delta'_{\mathrm{Sh}^\omega(\mathfrak{b})}$, we have also and $(\mathrm{id} \otimes S^{-1}_{\mathrm{Sh}^\omega(\mathfrak{b})})(\mathcal{R}^\omega(r_{\mathfrak{ab}})) = \mathcal{R}^\omega(r_{\mathfrak{ab}})^{-1}$. These equalities imply (7). $\qquad\square$

Remark 2. The sequence $(\mathcal{R}_n^\omega(r_{\mathfrak{ab}}))_{n \geq 0}$ is uniquely determined by the conditions that it satisfies (6), $\mathcal{R}_0^\omega(r_{\mathfrak{ab}}) = 1$, $\mathcal{R}_1^\omega(r_{\mathfrak{ab}}) = r_{\mathfrak{ab}}$, and $(\mathrm{pr}_{\mathfrak{a}} \otimes \mathrm{pr}_{\mathfrak{b}})(\mathcal{R}^\omega(r_{\mathfrak{ab}})) = 0$ for $n \geq 2$. It follows that $\mathcal{R}'_n(r_{\mathfrak{ab}})$ is equal to $\mathcal{R}'_n(r_{\mathfrak{ab}}^{(21)})^{(21)}$.

Remark 3. If the sum $\mathcal{R}^\omega(r_{\mathfrak{ab}}) = \sum_{i \geq 0} \mathcal{R}_i^\omega(r_{\mathfrak{ab}})$ makes sense (for example if we work with $\hbar$-adic completions and $r_{\mathfrak{ab}}$ is in $O(\hbar)$), it satisfies the R-matrix identities

$$(\Delta_{\mathrm{Sh}^\omega(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{b})})(\mathcal{R}^\omega(r_{\mathfrak{ab}})) = (\mathcal{R}^\omega)^{(13)}(r_{\mathfrak{ab}})(\mathcal{R}^\omega)^{(23)}(r_{\mathfrak{ab}}),$$

$$(\mathrm{id}_{\mathrm{Sh}(\mathfrak{a})} \otimes \Delta_{\mathrm{Sh}^\omega(\mathfrak{b})})(\mathcal{R}^\omega(r_{\mathfrak{ab}})) = (\mathcal{R}^\omega)^{(13)}(r_{\mathfrak{ab}})(\mathcal{R}^\omega)^{(12)}(r_{\mathfrak{ab}}).$$

Remark 4. If ω is ω' of Section 1.6, then the first $\mathcal{R}_i^\omega(r_{\mathfrak{ab}})$ are

$$\mathcal{R}_2^\omega(r_{\mathfrak{ab}}) = \sum_{i,j}(a_i a_j) \otimes (b_i b_j) + (a_i a_j) \otimes (b_j b_i) - \frac{1}{2}([a_i, a_j]) \otimes (b_i b_j) + \frac{1}{2}(a_i a_j) \otimes ([b_i, b_j])$$

$$= \sum_{i,j} \frac{1}{2}([a_i, a_j]) \otimes (b_j b_i) + (a_i a_j) \otimes (b_i)(b_j) = \sum_{i,j} \frac{1}{2}(a_i a_j) \otimes ([b_i, b_j]) + (a_i)(a_j) \otimes (b_j b_i)$$

$$\mathcal{R}_3^\omega(r_{\mathfrak{ab}}) = \sum_{i,j,k}(a_i a_j a_k) \otimes \big((b_i b_j b_k) + \text{all permutations in } i, j, k\big)$$

$$+ \frac{1}{2}(a_i a_j a_k) \otimes \big([b_i, b_j]b_k + b_k[b_i, b_j] + [b_i, b_k]b_j + b_j[b_i, b_k] + [b_j, b_k]b_i + b_i[b_j, b_k]\big)$$

$$+ \frac{1}{2}\big([a_i, a_j]a_k + a_k[a_i, a_j] + [a_i, a_k]a_j + a_j[a_i, a_k] + [a_j, a_k]a_i + a_i[a_j, a_k]\big) \otimes (b_k b_j b_i)$$

$$+ (a_i a_j a_k) \otimes (L_3(b_i, b_j, b_k)) + (L_3(a_i, a_j, a_k)) \otimes (b_k b_j b_i)$$

$$+ \frac{1}{4}([a_i, a_j]a_k) \otimes \big([b_j, b_k]b_i + b_j[b_i, b_k]\big) + \frac{1}{4}(a_i[a_j, a_k]) \otimes \big(b_k[b_i, b_j] + [b_i, b_k]b_j\big)$$

$$= \sum_{i,j,k}(a_i)(a_j)(a_k) \otimes (b_k b_j b_i) + \frac{1}{2}(a_i a_j)(a_k) \otimes (b_k[b_i, b_j]) + \frac{1}{2}(a_i)(a_j a_k) \otimes ([b_j, b_k]b_i)$$

$$+ (a_i a_j a_k) \otimes (L_3(b_i, b_j, b_k)),$$

where

$$L_3(x, y, z) = \frac{1}{6}\big([x, [y, z]] + [[x, y], z]\big).$$

3. CONSTRUCTION OF SOLUTIONS OF QYBE

In this Section, we again fix an element $\omega \in \mathcal{B}(\mathbb{K}[[\hbar]])$. Let us assume that the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ are the same, and that that we are given an element $r_\mathfrak{a} = \sum_{i \in I} a_i \otimes b_i$ of $\mathfrak{a} \otimes \mathfrak{a}$, satisfying the classical Yang-Baxter equation (CYBE), that is

$$[r_\mathfrak{a}^{(12)}, r_\mathfrak{a}^{(13)}] + [r_\mathfrak{a}^{(12)}, r_\mathfrak{a}^{(23)}] + [r_\mathfrak{a}^{(13)}, r_\mathfrak{a}^{(23)}] = 0.$$

Our aim in this Section is to construct a family $(\rho_n^\omega(r_\mathfrak{a}))_{n \geq 1}$ of elements of $\mathfrak{a} \otimes \mathfrak{a}$, expressed in terms of Lie polynomials of degree n in the a_i and b_i, such that if we set $\rho^\omega(r_\mathfrak{a}) = \sum_{n \geq 0} \rho_n^\omega(r_\mathfrak{a})$, and if we denote by $\mathcal{R}_i^\omega(\rho^\omega(r_\mathfrak{a}))[p]$ the homogeneous component of degree p in $r_\mathfrak{a}$ of $\mathcal{R}_i^\omega(\rho^\omega(r_\mathfrak{a}))$, then for each integer $n \geq 0$, $\rho^\omega(r_\mathfrak{a})$ satisfies the equation

$$\sum_{p,p',p'' | p+p'+p''=n} \sum_{i,j,k \geq 0} \mathcal{R}_i^\omega(\rho^\omega(r_\mathfrak{a}))[p]^{(12)} \mathcal{R}_j^\omega(\rho^\omega(r_\mathfrak{a}))[p']^{(13)} \mathcal{R}_k^\omega(\rho^\omega(r_\mathfrak{a}))[p'']^{(23)}$$

$$= \sum_{p,p',p'' | p+p'+p''=n} \sum_{i,j,k \geq 0} \mathcal{R}_k^\omega(\rho^\omega(r_\mathfrak{a}))[p'']^{(23)} \mathcal{R}_j^\omega(\rho^\omega(r_\mathfrak{a}))[p']^{(13)} \mathcal{R}_i^\omega(\rho^\omega(r_\mathfrak{a}))[p]^{(12)}.$$

In particular, if the sum $\sum_i \mathcal{R}_i^\omega(\rho^\omega(r_\mathfrak{a}))$ converges (for example if $r_\mathfrak{a}$ has positive $\hbar$-adic valuation), then $\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})) = \sum_{i \geq 0} \mathcal{R}_i^\omega(\rho^\omega(r_\mathfrak{a}))$ satisfies the quantum Yang-Baxter equation (QYBE)

$$(\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})))^{(12)} (\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})))^{(13)} (\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})))^{(23)}$$

$$= (\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})))^{(23)} (\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})))^{(13)} (\mathcal{R}^\omega(\rho^\omega(r_\mathfrak{a})))^{(12)}.$$

After we reduce this problem to a Lie algebraic problem (Section 3.1), we look for universal formulas expressing $\rho(r)$ and translate the Lie algebraic problem in terms of these universal formulas (Section 3.2). We then show that the latter problem can be formulated in cohomological terms (Section 3.3) and compute the relevant cohomologies (Section 3.4). We then gather our results to show the existence and unicity of universal formulas for $\rho^\omega(r_\mathfrak{a})$ (Section 3.5).

3.1. **Reduction to a Lie algebraic problem.** Let $\hbar$ be a formal variable and let ρ be an element of $\hbar(\mathfrak{a} \otimes \mathfrak{a})[[\hbar]]$. Then $\mathcal{R}^\omega(\rho)$ is an element of $\mathrm{Sh}(\mathfrak{a})^{\otimes 2}[[\hbar]]$. Replacing the ground field $\mathbb{K}$ by the ring $\mathbb{K}[[\hbar]]/(\hbar^n)$ in the definition of $\mathcal{R}^\omega$, we define a map $\mathcal{R}^\omega : \hbar \mathfrak{a}^{\otimes 2}[[\hbar]]/\hbar^n \mathfrak{a}^{\otimes 2}[[\hbar]] \to \mathrm{Sh}(\mathfrak{a})^{\otimes 2}[[\hbar]]/(\hbar^n)$. The following Proposition shows that the condition that $\mathcal{R}^\omega(\rho)$ be a solution of the associative QYBE is equivalent to ρ being a solution of equation (15), a Lie algebraic equation, which we call the Lie version of QYBE.

Proposition 3.1. *Let* $\mathrm{pr}_\mathfrak{a}$ *denote the projection of* $\mathrm{Sh}^\omega(\mathfrak{a})$ *on* $\mathfrak{a}$ *parallel to* $\oplus_{i \neq 1} \mathfrak{a}^{\otimes i}$. *Let* n *be an integer and* ρ_n *belong to* $\hbar \mathfrak{a}^{\otimes 2}[[\hbar]]/\hbar^n \mathfrak{a}^{\otimes 2}[[\hbar]]$. *Then the following statements are equivalent*
 i) $\mathcal{R}^\omega(\rho_n)$ *satisfies QYBE in* $\mathrm{Sh}^\omega(\mathfrak{a})^{\otimes 3}[[\hbar]]/(\hbar^n)$

ii) *the equation*

$$\mathrm{pr}_{\mathfrak{a}}^{\otimes 3}((\mathcal{R}^{\omega}(\rho_n))^{(12)}(\mathcal{R}^{\omega}(\rho_n))^{(13)}(\mathcal{R}^{\omega}(\rho_n))^{(23)}) \tag{12}$$
$$= \mathrm{pr}_{\mathfrak{a}}^{\otimes 3}((\mathcal{R}^{\omega}(\rho_n))^{(23)}(\mathcal{R}^{\omega}(\rho_n))^{(13)}(\mathcal{R}^{\omega}(\rho_n))^{(12)})$$

holds in $\mathfrak{a}^{\otimes 3}[[\hbar]]/(\hbar^n)$.

It follows that if ρ belongs to $\hbar\mathfrak{a}^{\otimes 2}[[\hbar]]$, the statements
iii) $\mathcal{R}^{\omega}(\rho)$ *satisfies QYBE in* $\mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 3}[[\hbar]]$
iv) *the equation*

$$\mathrm{pr}_{\mathfrak{a}}^{\otimes 3}((\mathcal{R}^{\omega}(\rho))^{(12)}(\mathcal{R}^{\omega}(\rho))^{(13)}(\mathcal{R}^{\omega}(\rho))^{(23)}) = \mathrm{pr}_{\mathfrak{a}}^{\otimes 3}((\mathcal{R}^{\omega}(\rho))^{(23)}(\mathcal{R}^{\omega}(\rho))^{(13)}(\mathcal{R}^{\omega}(\rho))^{(12)})$$
$$\tag{13}$$

holds in $\mathfrak{a}^{\otimes 3}[[\hbar]]$.

Proof. Let us prove the equivalence between *i)* and *ii)* by induction over n. The case $n = 0$ is trivial. Assume that the equivalence holds at order n and let us show it at order $n+1$. The direct sense is obvious. Let us assume now that ρ_{n+1} satisfies (12) at order $n+1$ and let us show that

$$Z = \mathcal{R}^{\omega}(\rho_{n+1})^{(12)}\mathcal{R}^{\omega}(\rho_{n+1})^{(13)}\mathcal{R}^{\omega}(\rho_{n+1})^{(23)} - \mathcal{R}^{\omega}(\rho_{n+1})^{(23)}\mathcal{R}^{\omega}(\rho_{n+1})^{(13)}\mathcal{R}^{\omega}(\rho_{n+1})^{(12)}$$

(an element of $\mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 3}[[\hbar]]/(\hbar^{n+1}))$ is zero. By the induction hypothesis, Z belongs to $\hbar^n \mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 3}[[\hbar]]/\hbar^{n+1}\mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes n+1}[[\hbar]]$. Set

$$L = \mathcal{R}^{\omega}(\rho_{n+1})^{(12)}\mathcal{R}^{\omega}(\rho_{n+1})^{(13)},$$

$$R = \mathcal{R}^{\omega}(\rho_{n+1})^{(23)}(\mathcal{R}^{\omega}(\rho_{n+1})^{(13)}\mathcal{R}^{\omega}(\rho_{n+1})^{(12)})(\mathcal{R}^{\omega}(\rho_{n+1})^{(23)})^{-1}.$$

Since $L - R = Z(\mathcal{R}^{\omega}(\rho_{n+1})^{(23)})^{-1}$ and $\mathcal{R}^{\omega}(\rho_{n+1})$ belongs to

$$1 + \hbar\,\mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 2}[[\hbar]]/\hbar^{n+1}\mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 2}[[\hbar]],$$

we get $L - R = Z$. On the other hand, both L and R satisfy

$$(\Delta_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id}^{\otimes 2})(L) = L^{(134)}L^{(234)} \quad \text{and} \quad (\Delta_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id}^{\otimes 2})(R) = R^{(134)}R^{(234)}. \tag{14}$$

Let us write $L = \sum_{i \geq 0} \hbar^i L_i$, $R = \sum_{i \geq 0} \hbar^i R_i$, with $L_i, R_i \in \mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 3}$. Then the induction hypothesis implies that $L_i = R_i$ for $i \leq n$ and (14) implies that

$$(\widetilde{\Delta}_{\mathrm{Sh}^{\omega}(\mathfrak{a})}\otimes\mathrm{id}^{\otimes 2})(L_{n+1}) = \sum_{i=1}^{n} L_i^{(134)}L_{n+1-i}^{(234)}, \quad (\widetilde{\Delta}^{\mathrm{Sh}^{\omega}(\mathfrak{a})}\otimes\mathrm{id}^{\otimes 2})(R_{n+1}) = \sum_{i=1}^{n} R_i^{(134)}R_{n+1-i}^{(234)},$$

where $\widetilde{\Delta}_{\mathrm{Sh}^{\omega}(\mathfrak{a})}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$. Therefore $(\widetilde{\Delta}_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id}^{\otimes 2})(L_{n+1}) = (\widetilde{\Delta}_{\mathrm{Sh}^{\omega}(\mathfrak{a})} \otimes \mathrm{id}^{\otimes 2})(R_{n+1})$, so $L_{n+1} - R_{n+1}$ belongs to $\mathfrak{a} \otimes \mathrm{Sh}^{\omega}(\mathfrak{a})^{\otimes 2}$. Let us define Z_{n+1} as $\hbar^{-n-1}Z$ mod $\hbar$. Then Z_{n+1}, then $Z_{n+1} = L_{n+1} - R_{n+1}$, therefore Z_{n+1} belongs to $\mathfrak{a} \otimes \mathrm{Sh}^{\omega}(\mathfrak{a}^{\otimes 2})$.

Applying the same reasoning to the comparison of

$$\mathcal{R}^{\omega}(\rho_{n+1})^{(13)}\mathcal{R}^{\omega}(\rho_{n+1})^{(23)} \text{ and } (\mathcal{R}^{\omega}(\rho_{n+1})^{(12)})^{-1}\big(\mathcal{R}^{\omega}(\rho_{n+1})^{(23)}\mathcal{R}^{\omega}(\rho_{n+1})^{(13)}\big)\mathcal{R}^{\omega}(\rho_{n+1})^{(12)},$$

resp., of

$$\mathcal{R}^\omega(\rho_{n+1})^{(12)}\mathcal{R}^\omega(\rho_{n+1})^{(23)} \quad \text{and} \quad \sum_\lambda B_\lambda^{(3)}\mathcal{R}^\omega(\rho_{n+1})^{(23)}\mathcal{R}^\omega(\rho_{n+1})^{(13)}\mathcal{R}(\rho_{n+1})^{(12)}A_\lambda^{(1)},$$

where $\sum_\lambda A_\lambda \otimes B_\lambda$ is the inverse of $\mathcal{R}^\omega(\rho_{n+1})^{-1}$ in $\mathrm{Sh}^\omega(\mathfrak{a}) \otimes \mathrm{Sh}^\omega(\mathfrak{a})^{opp}[[\hbar]]/(\hbar^{n+1})$, where $\mathrm{Sh}^\omega(\mathfrak{a})^{opp}$ is the opposite algebra of $\mathrm{Sh}^\omega(\mathfrak{a})$, we find that Z_{n+1} belongs to $\mathrm{Sh}^\omega(\mathfrak{a}^{\otimes 2}) \otimes \mathfrak{a}$, resp., to $\mathrm{Sh}^\omega(\mathfrak{a}) \otimes \mathfrak{a} \otimes \mathrm{Sh}^\omega(\mathfrak{a})$. Therefore, Z_{n+1} belongs to $\mathfrak{a}^{\otimes 3}$, so it may be identified with its image by $\mathrm{pr}_\mathfrak{a}^{\otimes 3}$. Since by assumption $\mathrm{pr}_\mathfrak{a}^{\otimes 3}(Z)$ is zero, we have $\mathrm{pr}_\mathfrak{a}^{\otimes 3}(Z_{n+1}) = 0$, therefore Z_{n+1} is equal to zero and Z is zero as well. This proves the induction step.

The equivalence between *iii)* and *iv)* follows easily from that of *i)* and *ii)*. $\square$

We will call equation (13) the *Lie version of QYBE* (Lie QYBE for short). This equation may be expressed as follows.

For $a_1, \dots, a_n$ elements of a Lie algebra $\mathfrak{g}$, let us denote by $L_n^\mathfrak{g}(a_1, \dots, a_n)$ the projection of the first summand of $\mathrm{Sh}^\omega(\mathfrak{g}) = \mathfrak{g} \oplus (\oplus_{i\neq 1}\mathfrak{g}^{\otimes i})$ of the product $(a_1)\cdots(a_n)$; so $L_n^\mathfrak{g}(a_1, \dots, a_n)$ belongs to $\mathfrak{g}$. When $n = 0$, we set $L_n^\mathfrak{g}(x_1, \dots, x_n) = 0$. There is a unique element of FL_n, which we denote by L_n, such that $L_n^\mathfrak{g}(a_1, \dots, a_n)$ is obtained from L_n by substituting a_i to the ith generator of FL_n, $i = 1, \dots, n$. L_n depends on ω. If $\underline{n} = (n_1, \dots, n_k)$, we set $\lambda_{\underline{n}} = \lambda_{\underline{n}}'^{(1)} \otimes \cdots \otimes \lambda_{\underline{n}}'^{(k)} \otimes \lambda_{\underline{n}}''$.

Then we have

$$\mathrm{pr}_\mathfrak{a}^{\otimes 3}\left(\mathcal{R}^\omega(\rho)^{(12)}\mathcal{R}^\omega(\rho)^{(13)}\mathcal{R}^\omega(\rho)^{(23)}\right) = \sum_{\xi',\xi'',\eta\geq 0}\sum_{\alpha_1,\dots,\alpha_\eta\geq 0}\sum_{n_1,\dots,n_{\xi'+\xi''}\geq 0, m_{11},\dots,m_{\eta\alpha_\eta}\geq 0}$$

$$\left(L_{\alpha_1+\cdots+\alpha_\eta+\xi'}(\lambda_{\underline{m}_1}'^{(1)}, \dots, \lambda_{\underline{m}_\eta}'^{(\alpha_\eta)}, \lambda_{\underline{n}}'^{(1)}, \dots, \lambda_{\underline{n}}'^{(\xi')}) \otimes \right.$$

$$\left. \otimes L_{\eta+\xi''}(\lambda_{\underline{m}_1}'', \dots, \lambda_{\underline{m}_\eta}'', \lambda_{\underline{n}}'^{(\xi'+1)}, \dots, \lambda_{\underline{n}}'^{(\xi'+\xi'')}) \otimes \lambda_{\underline{n}}''\right)(\rho),$$

where we set $\underline{n} = (n_i)_{i=1,\dots,\xi'+\xi''}$, $\underline{m}_\beta = (m_{\beta i})_{i=1,\dots,\alpha_\beta}$, for $\beta = 1, \dots, \eta$. In the same way,

$$\mathrm{pr}_\mathfrak{a}^{\otimes 3}\left(\mathcal{R}^\omega(\rho)^{(23)}\mathcal{R}^\omega(\rho)^{(13)}\mathcal{R}^\omega(\rho)^{(12)}\right) = \sum_{\xi',\xi'',\eta\geq 0}\sum_{\alpha_1,\dots,\alpha_\eta\geq 0}\sum_{n_1,\dots,n_{\xi'+\xi''}\geq 0, m_{11},\dots,m_{\eta\alpha_\eta}\geq 0}$$

$$\left(L_{\alpha_1+\cdots+\alpha_\eta+\xi''}(\lambda_{\underline{n}}'^{(\xi'+1)}, \dots, \lambda_{\underline{n}}'^{(\xi'+\xi'')}, \lambda_{\underline{m}_1}'^{(1)}, \dots, \lambda_{\underline{m}_\eta}'^{(\alpha_\eta)}) \otimes \right.$$

$$\left. \otimes L_{\eta+\xi'}(\lambda_{\underline{n}}'^{(1)}, \dots, \lambda_{\underline{n}}'^{(\xi')}, \lambda_{\underline{m}_1}'', \dots, \lambda_{\underline{m}_\eta}'') \otimes \lambda_{\underline{n}}''\right)(\rho).$$

So we have

Corollary 3.1. *The Lie QYB equation (13) is equivalent to the following equation*

$$\sum_{\xi',\xi'',\eta \geq 0} \sum_{\alpha_1,\dots,\alpha_\eta \geq 0} \sum_{n_1,\dots,n_{\xi'+\xi''} \geq 0, m_{11},\dots,m_{\eta\alpha_\eta} \geq 0}$$

$$\left(L_{\alpha_1+\cdots+\alpha_\eta+\xi'}(\lambda'^{(1)}_{\underline{m_1}},\dots,\lambda'^{(\alpha_\eta)}_{\underline{m_\eta}},\lambda'^{(1)}_{\underline{n}},\dots,\lambda'^{(\xi')}_{\underline{n}}) \otimes \right.$$

$$\left. \otimes L_{\eta+\xi''}(\lambda''_{\underline{m_1}},\dots,\lambda''_{\underline{m_\eta}},\lambda'^{(\xi'+1)}_{\underline{n}},\dots,\lambda'^{(\xi'+\xi'')}_{\underline{n}}) \otimes \lambda''_{\underline{n}} \right)(\rho)$$

$$= \sum_{\xi',\xi'',\eta \geq 0} \sum_{\alpha_1,\dots,\alpha_\eta \geq 0} \sum_{n_1,\dots,n_{\xi'+\xi''} \geq 0, m_{11},\dots,m_{\eta\alpha_\eta} \geq 0}$$

$$\left(L_{\alpha_1+\cdots+\alpha_\eta+\xi''}(\lambda'^{(\xi'+1)}_{\underline{n}},\dots,\lambda'^{(\xi'+\xi'')}_{\underline{n}},\lambda'^{(1)}_{\underline{m_1}},\dots,\lambda'^{(\alpha_\eta)}_{\underline{m_\eta}}) \otimes \right.$$

$$\left. \otimes L_{\eta+\xi'}(\lambda'^{(1)}_{\underline{n}},\dots,\lambda'^{(\xi')}_{\underline{n}},\lambda''_{\underline{m_1}},\dots,\lambda''_{\underline{m_\eta}}) \otimes \lambda''_{\underline{n}} \right)(\rho), \tag{15}$$

where ρ is an element of $\hbar(\mathfrak{a} \otimes \mathfrak{a})[[\hbar]]$.

3.2. **Universal formulation of the problem.** Let us explain more precisely the nature of the function $r_{\mathfrak{a}} \mapsto \rho_n(r_{\mathfrak{a}})$.

If n is an integer, let us denote by $Free_n$ the part of the free Lie algebra with n generators, homogeneous of degree one in each generator. The action of $\mathfrak{S}_n$ by permutation of the generators induces a $\mathfrak{S}_n$-module structure on $Free_n$.

If Γ is a group acting on a vector space M, we denote by M_Γ the space of coinvariants of M with respect to Γ; it is defined as $M_\Gamma = M/\operatorname{Span}\{\gamma m - m, \gamma \in \Gamma, m \in M\}$.

Lemma 3.1. *Let us set $F_n = (Free_n \otimes Free_n)_{\mathfrak{S}_n}$. Let $(\mathfrak{a}, r_{\mathfrak{a}})$ be the pair of a Lie algebra and an element $r_{\mathfrak{a}} = \sum_{i\in I} a_i \otimes b_i \in \mathfrak{a} \otimes \mathfrak{a}$ satisfying of CYBE. There is a unique map $\kappa_{r_{\mathfrak{a}}}^{(ab)} : F_n \to \mathfrak{a} \otimes \mathfrak{a}$, such that if $P(x_1,\dots,x_n)$ and $Q(x_1,\dots,x_n)$ belong to $Free_n$, then the image by $\kappa_{r_{\mathfrak{a}}}^{(ab)}$ of the class of $P \otimes Q$ is*

$$\sum_{i_1,\dots,i_n \in I} P(a_{i_1},\dots,a_{i_n}) \otimes Q(b_{i_1},\dots,b_{i_n}).$$

Proof. This follows from the fact that for any σ in $\mathfrak{S}_n$, we have

$$\sum_{i_1,\dots,i_n \in I} P(a_{i_1},\dots,a_{i_n}) \otimes Q(b_{i_1},\dots,b_{i_n}) = \sum_{i_1,\dots,i_n \in I} P(a_{i_{\sigma(1)}},\dots,a_{i_{\sigma(n)}}) \otimes Q(b_{i_{\sigma(1)}},\dots,b_{i_{\sigma(n)}}).$$

$\square$

We will introduce equations for a system of elements $(\varrho_n)_{n\geq 1}$, where ϱ_n belongs to F_n, which we will call the universal Lie QYB equations.

The universal Lie QYB equations have the following property: if $\mathfrak{a}$ is a Lie algebra and $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution of CYBE, if $(\varrho_n)_{n\geq 1} \in \prod_{n\geq 1} F_n$ is a solution of the universal Lie QYB equations, and if we set $\rho_n(r_{\mathfrak{a}}) = \kappa_{r_{\mathfrak{a}}}^{(ab)}(\varrho_n)$, then the family $(\rho_n(r_{\mathfrak{a}}))_{n\geq 0}$ satisfies the Lie QYB equations in $\mathfrak{a}^{\otimes 3}$.

After we establish this fact, we will show that the universal Lie QYB equations have a unique solution such that $\varrho_1 = x_1 \otimes y_1$.

3.2.1. *Insertion map.* Let p, q, r be integers. Denote by $FL_{p,q,r}$ the space

$$FL_{p,q,r} = \left(FL_{q+r} \otimes FL_{p+r} \otimes FL_{p+q} \right)_{\mathfrak{S}_p \times \mathfrak{S}_q \times \mathfrak{S}_r};$$

here FL_{q+r} (resp., FL_{p+r}, FL_{p+q}) is generated by the variables $v_1, \ldots, v_q, w_1, \ldots, w_r$ (resp., by $u_1, \ldots, u_p, w_1', \ldots, w_r'$ and by $u_1', \ldots, u_p', v_1', \ldots, v_q'$), and $\mathfrak{S}_p$, $\mathfrak{S}_q$ and $\mathfrak{S}_r$ acts by simultaneously permuting the variables x_i and x_i' $(x = u, v, w)$.

Then there is a unique linear map

$$\mathrm{ins} : FL_{p,q,r} \otimes \left(\prod_{n \geq 1} F_n \right) \to \prod_{p',q',r' \geq 0} FL_{p',q',r'},$$

such that the (p', q', r') component of $\mathrm{ins}\left((P \otimes Q \otimes R) \otimes \left(\sum_{n \geq 1} \sigma_n \right) \right)$ is

$$\sum_{(n_i),(m_i),(s_i) \mid \sum_{i=1}^p n_i = p', \sum_{i=1}^p m_i = q', \sum_{i=1}^p s_i = r'} P(\widetilde{\sigma}'_{m_1}, \ldots, \widetilde{\sigma}'_{m_q}, \widetilde{\sigma}'_{s_1}, \ldots, \widetilde{\sigma}'_{s_r})$$

$$\otimes Q(\widetilde{\sigma}'_{n_1}, \ldots, \widetilde{\sigma}'_{n_p}, \widetilde{\sigma}''_{s_1}, \ldots, \widetilde{\sigma}''_{s_r}) \otimes R(\widetilde{\sigma}''_{n_1}, \ldots, \widetilde{\sigma}''_{n_p}, \widetilde{\sigma}''_{m_1}, \ldots, \widetilde{\sigma}''_{m_q}),$$

where we set $\sigma_n = \sigma_n' \otimes \sigma_n''$ and $\widetilde{\sigma}'_{m_i} = \sigma'_{m_i}(v_{m_1 + \cdots + m_{i-1}+1}, \ldots, v_{m_1 + \cdots + m_i})$, $\widetilde{\sigma}''_{m_i} = \sigma''_{m_i}(v'_{m_1 + \cdots + m_{i-1}+1}, \ldots, v'_{m_1 + \cdots + m_i})$, $\widetilde{\sigma}'_{n_i} = \sigma'_{n_i}(u_{n_1 + \cdots + n_{i-1}+1}, \ldots, u_{n_1 + \cdots + n_i})$, $\widetilde{\sigma}''_{n_i} = \sigma''_{n_i}(u'_{n_1 + \cdots + n_{i-1}+1}, \ldots, u'_{n_1 + \cdots + n_i})$, and $\widetilde{\sigma}'_{s_i} = \sigma'_{s_i}(w_{s_1 + \cdots + s_{i-1}+1}, \ldots, w_{s_1 + \cdots + s_i})$, $\widetilde{\sigma}''_{s_i} = \sigma''_{s_i}(w'_{s_1 + \cdots + s_{i-1}+1}, \ldots, w'_{s_1 + \cdots + w_i})$.

3.2.2. *Universal version of the Lie QYBE identity.* When N is an integer ≥ 0, let us denote by $F_N^{(aab)}$ and $F_N^{(abb)}$ the direct sums

$$F_N^{(aab)} = \oplus_{p,q \geq 1 \mid p+q=N} (FL_p \otimes FL_q \otimes FL_N)_{\mathfrak{S}_p \times \mathfrak{S}_q},$$

$$F_N^{(abb)} = \oplus_{p,q \geq 1 \mid p+q=N} (FL_N \otimes FL_p \otimes FL_q)_{\mathfrak{S}_p \times \mathfrak{S}_q},$$

where the $Free_N$ (resp., $Free_p$ and $Free_q$) is endowed with the action of $\mathfrak{S}_p \times \mathfrak{S}_q$ provided by the inclusion map $\mathfrak{S}_p \times \mathfrak{S}_q \to \mathfrak{S}_N$ (resp., the projection map of $\mathfrak{S}_p \times \mathfrak{S}_q$ on $\mathfrak{S}_p$ and $\mathfrak{S}_q$). So $F_N^{(aab)} = \oplus_{p,q \mid p+q=N} FL_{p,q,0}$ and $F_N^{(abb)} = \oplus_{p,q \mid p+q=N} FL_{0,p,q}$.

In the same way as Lemma 3.1, we have

Lemma 3.2. *There are unique linear maps $\kappa_r^{(aab)} : F_N^{(aab)} \to \mathfrak{a}^{\otimes 3}$ and $\kappa_r^{(abb)} : F_N^{(abb)} \to \mathfrak{a}^{\otimes 3}$, such that for $P \in Free_N$ and P', P'' in $Free_p$ and $Free_q$, the image by $\kappa_N^{(abb)}$ of the class of $P \otimes P' \otimes P''$ is*

$$\sum_{i_1, \ldots, i_N \in I} P(a_{i_1}, \ldots, a_{i_N}) \otimes P'(b_{i_1}, \ldots, b_{i_p}) \otimes P''(b_{i_{p+1}}, \ldots, b_{i_N})$$

and the image by $\kappa_N^{(aab)}$ of the class of $P' \otimes P'' \otimes P$ is

$$\sum_{i_1, \ldots, i_N \in I} P'(a_{i_1}, \ldots, a_{i_p}) \otimes P''(a_{i_{p+1}}, \ldots, a_{i_N}) \otimes P(b_{i_1}, \ldots, b_{i_N}).$$

Proposition 3.2. *If p, q, r are integers, there is a unique map $\mu_{Lie}^{p,q,r} : FL_{p,q,r} \rightarrow$ $F_{p+q+r}^{(aab)} \oplus F_{p+q+r}^{(abb)}$, such that if $(\mathfrak{a}, r_{\mathfrak{a}})$ is the pair of a Lie algebra and a solution $r_{\mathfrak{a}} = \sum_{i \in I} a_i \otimes b_i \in \mathfrak{a} \otimes \mathfrak{a}$ of CYBE, then*

$$(\kappa_{p+q+r}^{(aab)} \oplus \kappa_{p+q+r}^{(abb)})(\mu_{Lie}^{p,q,r}(P \otimes Q \otimes R)) = \sum_{i_1,\dots,k_r \in I} P(a_{j_1}, \dots, a_{j_q}, a_{k_1}, \dots, a_{k_r})$$

$$\otimes Q(a_{i_1}, \dots, a_{i_p}, b_{k_1}, \dots, b_{k_r}) \otimes R(b_{i_1}, \dots, b_{i_p}, b_{j_1}, \dots, b_{j_q})$$

We define μ_{Lie} as the direct sum $\oplus_{p,q,r}\mu_{Lie}^{p,q,r}$.

Proof. See Appendix B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let $(\varrho_n)_{n \geq 1}$ be an element of $\prod_{n \geq 1} F_n$. We say that $(\varrho_n)_{n \geq 1}$ is a solution of the *universal Lie QYB equations* if for any integer N, the equality

$$\sum_{\xi',\xi'',\eta \geq 0} \sum_{\alpha_1,\dots,\alpha_\eta \geq 0} \sum_{n_1,\dots,n_{\xi'+\xi''} \geq 0, m_{11},\dots,m_{\eta\alpha_\eta} \geq 0}$$

$$\mu_{Lie} \circ \text{ins} \left(\left(L_{\alpha_1+\dots+\alpha_\eta+\xi'}(\lambda_{\underline{m}_1}'^{(1)}, \dots, \lambda_{\underline{m}_\eta}'^{(\alpha_\eta)}, \lambda_{\underline{n}}'^{(1)}, \dots, \lambda_{\underline{n}}'^{(\xi')}) \otimes \right. \right.$$

$$\left. \left. \otimes L_{\eta+\xi''}(\lambda_{\underline{m}_1}'', \dots, \lambda_{\underline{m}_\eta}'', \lambda_{\underline{n}}'^{(\xi'+1)}, \dots, \lambda_{\underline{n}}'^{(\xi'+\xi'')}) \otimes \lambda_{\underline{n}}'' \right) \otimes \left(\sum_n \varrho_n \right) \right)_N$$

$$= \sum_{\xi',\xi'',\eta \geq 0} \sum_{\alpha_1,\dots,\alpha_\eta \geq 0} \sum_{n_1,\dots,n_{\xi'+\xi''} \geq 0, m_{11},\dots,m_{\eta\alpha_\eta} \geq 0}$$

$$\mu_{Lie} \circ \text{ins} \left(\left(L_{\alpha_1+\dots+\alpha_\eta+\xi''}(\lambda_{\underline{n}}'^{(\xi'+1)}, \dots, \lambda_{\underline{n}}'^{(\xi'+\xi'')}, \lambda_{\underline{m}_1}'^{(1)}, \dots, \lambda_{\underline{m}_\eta}'^{(\alpha_\eta)}) \otimes \right. \right.$$

$$\left. \left. \otimes L_{\eta+\xi'}(\lambda_{\underline{n}}'^{(1)}, \dots, \lambda_{\underline{n}}'^{(\xi')}, \lambda_{\underline{m}_1}'', \dots, \lambda_{\underline{m}_\eta}'') \otimes \lambda_{\underline{n}}'' \right) \otimes \left(\sum_n \varrho_n \right) \right)_N, \qquad (16)$$

holds, where the index N means the homogeneous component in $F_N^{(aab)} \oplus F_N^{(abb)}$. We call the above equation the universal Lie QYB equation of degree N. We supplement this equation by the condition $\varrho_1 = x \otimes x$, where x is the canonical generator of $Free_1$.

Then we have

Lemma 3.3. *Assume that $(\varrho_n)_{n \geq 1}$ is a solution of the universal Lie QYB equations. Let $\mathfrak{a}$ be any Lie algebra and let $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ be a solution of CYBE. Set $\rho_n = \kappa_{r_{\mathfrak{a}}}^{(ab)}(\varrho_n)$. Then $\sum_{n \geq 1} \hbar^n \rho_n$ is a solution of the Lie QYB equation (15) in $\mathfrak{a}^{\otimes 3}[[\hbar]]$.*

Proof. This follows at once from Proposition 3.2. $\qquad\qquad\qquad\qquad\qquad$ $\square$

3.3. **Cohomological formulation.** Our aim is to solve equations (16): we will show that these equations have a unique solution. For this, we will formulate equations (16) in cohomological terms.

For any integer n, there is a unique linear map $\delta_3^{(F)} : F_n \to F_{n+1}^{(aab)} + F_{n+1}^{(abb)}$, such that for any P and Q in FL_n,

$$\delta_3^{(F)}(P \otimes Q) = [w_1, P(v_1, \dots, v_n)] \otimes w_1' \otimes Q(v_1', \dots, v_n')$$
$$+ \mu_{Lie}^{1,0,n}(w_1 \otimes [w_1', P(u_1, \dots, u_n)] \otimes Q(u_1', \dots, u_n')) + v_1 \otimes P(u_1, \dots, u_n) \otimes [v_1', Q(u_1', \dots, u_n')]$$
$$+ [P(w_1, \dots, w_n), v_1] \otimes v_1' \otimes Q(w_1', \dots, w_n') + \mu_{Lie}^{1,0,n}(P(w_1, \dots, w_n) \otimes [Q(w_1', \dots, w_n'), u_1] \otimes u_1'$$
$$+ P(v_1, \dots, v_n) \otimes u_1 \otimes [Q(v_1', \dots, v_n'), u_1'].$$

Lemma 3.4. *Let $\mathfrak{a}$ be any Lie algebra and $r_\mathfrak{a} \in \mathfrak{a} \otimes \mathfrak{a}$ be a solution of CYBE. Let us define $\delta_{3,r_\mathfrak{a}} : \mathfrak{a}^{\otimes 2} \to \mathfrak{a}^{\otimes 3}$ by $\delta_{3,r_\mathfrak{a}}(x) = [r_\mathfrak{a}^{12}, x^{13}] + [r_\mathfrak{a}^{12}, x^{23}] + [r_\mathfrak{a}^{13}, x^{23}] + [x^{12}, r_\mathfrak{a}^{13}] + [x^{12}, r_\mathfrak{a}^{23}] + [x^{13}, r_\mathfrak{a}^{23}]$. Then the diagram*

$$
\begin{array}{ccc}
F_n & \xrightarrow{\delta_3^{(F)}} & F_{n+1}^{(aab)} \oplus F_{n+1}^{(abb)} \\
\downarrow \kappa_{r_\mathfrak{a}} & & \downarrow \kappa_{r_\mathfrak{a}}^{(aab)} \oplus \kappa_r^{(abb)} \\
\mathfrak{a}^{\otimes 2} & \xrightarrow{\delta_{3,r_\mathfrak{a}}} & \mathfrak{a}^{\otimes 3}
\end{array}
$$

is commutative.

Proof. This follows from Proposition 3.2. $\qquad\square$

Then there is a unique map $\Phi_N : \prod_{i=2}^{N-2} F_i \to F_N^{(aab)} \oplus F_N^{(abb)}$, which is polynomial of degree 3, such that the left side of equation (16) is equal to $\delta_3^{(F)}(\varrho_{N-1}) + \Phi_N(\varrho_2, \dots, \varrho_{N-2})$ (recall that ϱ_1 is the canonical generator of F_1). Equation (16) may then be rewritten as follows

$$\delta_3^{(F)}(\varrho_{N-1}) + \Phi_N(\varrho_2, \dots, \varrho_{N-2}) = 0. \tag{17}$$

If $\mathfrak{a}$ is a Lie algebra and $r_\mathfrak{a} \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution of CYBE, let us define $\delta_{4,r_\mathfrak{a}}$ as the linear map from $\mathfrak{a}^{\otimes 3}$ to $\mathfrak{a}^{\otimes 4}$ such that for any $x \in \mathfrak{a}^{\otimes 3}$,

$$\delta_{4,r_\mathfrak{a}}(x) = [r_\mathfrak{a}^{12} + r_\mathfrak{a}^{13} + r_\mathfrak{a}^{14}, x^{234}] - [-r_\mathfrak{a}^{12} + r_\mathfrak{a}^{23} + r_\mathfrak{a}^{24}, x^{134}]$$
$$+ [-r_\mathfrak{a}^{13} - r_\mathfrak{a}^{23} + r_\mathfrak{a}^{34}, x^{124}] - [-r_\mathfrak{a}^{14} - r_\mathfrak{a}^{24} - r_\mathfrak{a}^{34}, x^{123}].$$

In the same way as $\delta_3^{(F)}$ is universal version of $\delta_{3,r_\mathfrak{a}}$, we associate to $\delta_{4,r_\mathfrak{a}}$ its universal version $\delta_4^{(F)}$. For this, let us first define the spaces $F_N^{(x_1,\dots,x_n)}$. If n is an integer ≥ 1 and $(x_1, \dots, x_n)$ is a sequence of $\{a,b\}^n$, we define $F_N^{(x_1,\dots,x_n)}$ as follows. Let us set $K = \{k|x_k = a\}$ and $L = \{l|x_l = b\}$. K and L therefore form a partition of $\{1, \dots, n\}$. If $(p_{kl})_{(k,l) \in K \times L}$ belongs to $\mathbb{N}^{K \times L}$, let us set $f((p_{kl}), i) = \sum_{l \in L} p_{il}$ if $i \in K$ and $f((p_{kl}), i) = \sum_{k \in K} p_{ki}$ if $i \in L$. We then set

$$F_{(p_{kl})_{(k,l) \in K \times L}}^{(x_1,\dots,x_n)} = \left(\bigotimes_{i=1}^{n} Free_{f((p_{kl}),i)} \right)_{\prod_{k \in K, l \in L} \mathfrak{S}_{p_{kl}}}, \tag{18}$$

where each $\mathfrak{S}_{p_{kl}}$ acts by simultaneously permuting the generators of index $\sum_{j=1}^{l-1} p_{kj} + 1, \ldots, \sum_{j=1}^{l} p_{kj}$ of the kth tensor factor, and the generators of index $\sum_{i=1}^{k-1} p_{il} + 1, \ldots, \sum_{i=1}^{k} p_{il}$ of the lth tensor factor.

$$F^{(x_1,\ldots,x_n)} = \bigoplus_{(p_{kl})\in\mathbb{N}^{K\times L}|p_{kl}=0 \ \text{if}\ k>l} \left(\bigotimes_{i=1}^{n} Free_{f((p_{kl}),i)} \right)_{\prod_{k\in K,l\in L} \mathfrak{S}_{p_{kl}}} \tag{19}$$

and

$$F_N^{(x_1,\ldots,x_n)} = \bigoplus_{(p_{kl})\in\mathbb{N}^{K\times L}|p_{kl}=0 \ \text{if}\ k>l \ \text{and}\ \sum_{k\in K,l\in L} p_{kl}=N} \left(\bigotimes_{i=1}^{n} Free_{f((p_{kl}),i)} \right)_{\prod_{k\in K,l\in L} \mathfrak{S}_{p_{kl}}}.$$

For example,

$$F_N^{(aaab)} = \bigoplus_{p,p',p''|p+p'+p''=N} \left(Free_p \otimes Free_{p'} \otimes Free_{p''} \otimes Free_N \right)_{\mathfrak{S}_p \times \mathfrak{S}_{p'} \times \mathfrak{S}_{p''}}$$

and

$$F_N^{(aabb)} = \bigoplus_{p,p',q,q'|p+p'+q+q'=N} \left(Free_{p+p'} \otimes Free_{q+q'} \otimes Free_{p+q} \otimes Free_{p'+q'} \right)_{\mathfrak{S}_p \times \mathfrak{S}_{p'} \times \mathfrak{S}_q \times \mathfrak{S}_{q'}},$$

where in the last equality, $\mathfrak{S}_p$ acts by permutation of the p first generators of $Free_{p+p'}$ and $Free_{p+q}$, $\mathfrak{S}_{p'}$ acts by permutation of the p' last (resp., first) generators of $Free_{p+p'}$ (resp., $Free_{p'+q'}$), $\mathfrak{S}_q$ acts by permutation of the q first (resp., last) generators of $Free_{q+q'}$ (resp., $Free_{p+q}$), and $\mathfrak{S}_{q'}$ acts by permutation of the q' last generators of $Free_{q+q'}$ and $Free_{p'+q'}$.

As before, we associate to any Lie algebra $\mathfrak{a}$ and any element $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$, the map

$$\kappa_{r_{\mathfrak{a}}}^{(x_1,\ldots,x_n)} : F^{(x_1\ldots x_n)} \to \mathfrak{a}^{\otimes n}$$

such that if $(p_{kl}) \in \mathbb{N}^{K\times L}$ and we set $N = \sum_{k\in K, l\in L} p_{kl}$, and if $P_\alpha \in Free_{f((p_{kl}),\alpha)}$, then

$$\kappa_{r_{\mathfrak{a}}}^{(x_1,\ldots,x_n)}(\otimes_{i=1}^n P_i) = \sum_{\alpha(1),\ldots,\alpha(N)\in I} \bigotimes_{i=1}^{n} A_i,$$

where $A_i = P_i(a_{\alpha(p_1+\cdots+p_{i-1})+1}, \ldots, a_{\alpha(p_1+\cdots+p_i)})$ if $i \in K$ and

$$A_i = P_i(b_{\beta(q_1+\cdots+q_{i-1})+1}, \ldots, b_{\beta(q_1+\cdots+q_i)})$$

if $i \in L$, and we set $p_k = \sum_{l\in L} p_{kl}$ if $k \in K$, $q_l = \sum_{k\in K} p_{kl}$ if $l \in L$, and $\beta(q_1+\cdots+q_{i-1}+p_{1i}+\cdots+p_{k-1,j}+s) = \alpha(p_1+\cdots+p_{i-1}+p_{i1}+\cdots+p_{i,j-1}+s)$ if $1 \leq s \leq p_{kj}$.

Proposition 3.3. *There exists a map*

$$\delta_4^{(F)} : \bigoplus_{x\in\{a,b\}} F_n^{(axb)} \to \bigoplus_{x,y\in\{a,b\}} F_{n+1}^{(axyb)}$$

such that for any pair $(\mathfrak{a}, r_\mathfrak{a})$ *of a Lie algebra and a solution of CYBE, the diagram*

$$
\begin{array}{ccc}
\bigoplus_{x\in\{a,b\}} F_n^{(axb)} & \overset{\delta_4^{(F)}}{\longrightarrow} & \bigoplus_{x,y\in\{a,b\}} F_{n+1}^{(axyb)} \\
\downarrow \oplus_{x\in\{a,b\}}\kappa_{r_\mathfrak{a}}^{(axb)} & & \downarrow \oplus_{x,y\in\{a,b\}}\kappa_{r_\mathfrak{a}}^{(axyb)} \\
\mathfrak{a}^{\otimes 3} & \overset{\delta_{4,r_\mathfrak{a}}}{\longrightarrow} & \mathfrak{a}^{\otimes 4}
\end{array}
$$

is commutative.

Proof. See Appendix C. □

Proposition 3.4. *1) If* $\mathfrak{a}$ *is any Lie algebra and* $r_\mathfrak{a} \in \mathfrak{a} \otimes \mathfrak{a}$ *is any solution of CYBE, we have* $\delta_{4,r_\mathfrak{a}} \circ \delta_{3,r_\mathfrak{a}} = 0$.
   2) We have also $\delta_4^{(F)} \circ \delta_3^{(F)} = 0$.

Proof. 1) is a direct computation. The proof of 2) is in Appendix C. □

Remark 5. When $r_\mathfrak{a} + r_\mathfrak{a}^{(21)}$ is $\mathfrak{a}$-invariant, $\delta_{3,r_\mathfrak{a}}$ and $\delta_{4,r_\mathfrak{a}}$ are differentials of the Lie coalgebra cohomology complex of $\mathfrak{a}$, endowed with the Lie coalgebra structure given by $\delta(x) = [r_\mathfrak{a}, x \otimes 1 + 1 \otimes x]$. This explains the relation $\delta_{4,r_\mathfrak{a}} \circ \delta_{3,r_\mathfrak{a}} = 0$. On the other hand, $\delta_3^{(F)}$ and $\delta_4^{(F)}$ are also differentials of a complex whose degree n part is

$$\bigoplus_{x_1,\dots,x_{n-2}\in\{a,b\}} F^{(ax_1\dots x_{n-2}b)},$$

which may be viewed as a universal version of the Lie coalgebra cohomology complex of $\mathfrak{a}$. □

Theorem 3.1. *Let* $\mathfrak{a}$ *be a Lie algebra and* $\rho_1, \dots, \rho_N$ *be elements of* $\mathfrak{a}^{\otimes 2}$. *Set* $r = \rho_1$ *and assume that* $\rho_1, \dots, \rho_{N-1}$ *satisfy the equations*

$$\delta_{3,r}(\rho_{M-1}) + \Phi_M(\rho_2, \dots, \rho_{M-2}) = 0 \tag{20}$$

for $M = 1, \dots, N$. *(In particular, r is a solution of CYBE.) Then*

$$\delta_{4,r}(\Phi_{N+1}(\rho_2, \dots, \rho_{N-1})) = 0.$$

Proof. The proof of this Theorem relies on the following Proposition, which will be proved in Appendix A.

Proposition 3.5. *Let* A *be an algebra and* r_A *belong to* $A \otimes A$. *Let* N *be an integer* ≥ 3 *and assume that* $\mathcal{R}_1, \dots, \mathcal{R}_{N-2}$ *in* $A \otimes A$ *satisfy* $\mathcal{R}_1 = r_A$ *and*

$$[\![r_A, \mathcal{R}_i]\!] = - \sum_{p+q+r=i+1,p,q,r>0} \mathcal{R}_p^{12}\mathcal{R}_q^{13}\mathcal{R}_r^{23} + \sum_{p+q+r=i+1,p,q,r>0} \mathcal{R}_r^{23}\mathcal{R}_q^{13}\mathcal{R}_p^{12}, \tag{21}$$

for $i = 1, \dots, N-2$ *(in particular, r_A is a solution of CYBE). Then*

$$\delta(r_A | \sum_{p+q+r=N,p,q,r>0} \mathcal{R}_p^{12}\mathcal{R}_q^{13}\mathcal{R}_r^{23} - \mathcal{R}_r^{23}\mathcal{R}_q^{13}\mathcal{R}_p^{12}) \tag{22}$$

is zero. Here we set, if $a, b \in A$,

$$[\![a, b]\!] = [a^{12}, b^{13}] + [a^{12}, b^{23}] + [a^{13}, b^{23}] + [b^{12}, a^{13}] + [b^{12}, a^{23}] + [b^{13}, a^{23}],$$

and if $T \in A^{\otimes 3}$,

$$\delta(a|T) = [a^{12} + a^{13} + a^{14}, T^{234}] - [-a^{12} + a^{23} + a^{24}, T^{134}]$$
$$+ [-a^{13} - a^{23} + a^{34}, T^{124}] + [a^{14} + a^{24} + a^{34}, T^{123}].$$

Let us now prove Theorem 3.1. Let us define $\rho^{(N)}$ as the element $\sum_{i=1}^{N} \hbar^i \rho_i$ of $\hbar \mathfrak{a}^{\otimes 2}[[\hbar]]/\hbar^{N+1}\mathfrak{a}^{\otimes 2}[[\hbar]]$. Then $\mathcal{R}^{\omega}(\rho^{(N)})$ is an element of $\mathrm{Sh}^{\omega}(\mathfrak{a})^{\widehat{\otimes}3}/(\hbar^{N+1})$, and by Proposition 3.1, it satisfies QYBE. Let us expand $\mathcal{R}^{\omega}(\rho^{(N)})$ as $\sum_{i=1}^{N} \hbar^i \mathcal{R}_i$. We may apply Theorem 3.1 with $A = \mathrm{Sh}(\mathfrak{a})$ and $N-2$ replaced by N. Then the conclusion of this Theorem says that $\delta(r|\sum_{i,j,k>0, i+j+k=N+1} \mathcal{R}_i^{12}\mathcal{R}_j^{13}\mathcal{R}_k^{23} - \mathcal{R}_k^{23}\mathcal{R}_j^{13}\mathcal{R}_i^{12})$ is zero. To apply $\mathrm{pr}_{\mathfrak{a}}^{\otimes 4}$ to this identity, we use the following Lemma. Let us denote by ι the natural embedding of $\mathfrak{a}$ in $\mathrm{Sh}(\mathfrak{a})$, sending x to (x).

Lemma 3.5. *If $a \in \mathfrak{a}$ and $T \in \mathrm{Sh}(\mathfrak{a})$, then $\mathrm{pr}_{\mathfrak{a}}([\iota(a), T]) = [a, \mathrm{pr}_{\mathfrak{a}}(T)]$.*

Proof. We may assume that T is homogeneous. The statement is obvious when T has degree ≤ 1. Assume that $T = (x_1 \dots x_n)$ with $n \geq 2$, then $\mathrm{pr}_{\mathfrak{a}}([\iota(a), T]) = B_{1,n}(a|x_1 \dots x_n) - B_{n,1}(x_1 \dots x_n|a)$. Set $\beta_n(a|x_1 \dots x_n) = B_{1,n}(a|x_1, \dots, x_n) - B_{n,1}(x_1, \dots, x_n|a)$. Then β_n is an element of FL_{n+1} and we should prove that $\beta_n(a|x_1 \dots x_n)$ is identically zero. Let us prove this by induction on n. Assume that we have shown that when $m < n$, $\beta_m(a|x_1 \dots x_m) = 0$. Then we have

$$[(a), (x_1 \dots x_n)] = \sum_{i=1}^{n} (x_1, \dots [a, x_i] \dots x_n) + (\beta_n(a|x_1 \dots x_n)).$$

Then the Jacobi identity implies that

$$\left([a', \beta_n(a|x_1, \dots, x_n)] + \sum_{i=1}^{n} \beta_n(a'|x_1, \dots, [a, x_i], \dots, x_n)\right) \qquad (23)$$

$$- \left([a, \beta_n(a'|x_1, \dots, x_n)] + \sum_{i=1}^{n} \beta_n(a|x_1, \dots, [a', x_i], \dots, x_n)\right) \qquad (24)$$

$$= \beta_n([a', a]|x_1, \dots, x_n); \qquad (25)$$

since β_n is linear in each argument, this implies that $\beta_n([a', a]|x_1, \dots, x_n) = 0$. This is a universal identity, valid in FL_{n+2}. Therefore the element β_n of FL_{n+1} is zero. $\qquad \square$

We have already seen that $\mathrm{pr}_{\mathfrak{a}}^{\otimes 4}\left(\delta(r|\sum_{i,j,k>0, i+j+k=N+1} \mathcal{R}_i^{12}\mathcal{R}_j^{13}\mathcal{R}_k^{23} - \mathcal{R}_k^{23}\mathcal{R}_j^{13}\mathcal{R}_i^{12})\right)$ is zero. It follows from Lemma 3.5 that this is the image by $\delta_{4,r}$ of

$$\mathrm{pr}_{\mathfrak{a}}^{\otimes 3}\left(\sum_{i,j,k>0, i+j+k=N+1} \mathcal{R}_i^{12}\mathcal{R}_j^{13}\mathcal{R}_k^{23} - \mathcal{R}_k^{23}\mathcal{R}_j^{13}\mathcal{R}_i^{12}\right),$$

which is equal to $\Phi_{N+1}(\rho_2, \dots, \rho_{N-1})$. Therefore the image of $\Phi_{N+1}(\rho_2, \dots, \rho_{N-1})$ by $\delta_{4,r}$ is zero. This proves Theorem 3.1. $\qquad\square$

Theorem 3.1 has the following "universal" counterpart.

Theorem 3.2. *Assume that p is an integer and $\varrho_1, \dots, \varrho_p$ belong to $F_1, \dots, F_p$ and satisfy the universal Lie QYB equations of order $\leq p$ (this is the system of equations (16), where N takes the values $1, \dots, p$). Then the image by $\delta_4^{(F)}$ of the right side of equation (17), in which N takes the value $p + 1$, is zero.*

Proof. See Appendix E. $\qquad\square$

3.4. Cohomology groups H_n^2 and H_n^3.

3.4.1. *Definition of H_n^2 and H_n^3.* When n and N are integers, let us set

$$F_N^{Lie,(n)} = \bigoplus_{x_1, \dots, x_{n-2} \in \{a,b\}} F_N^{(ax_1 \dots x_{n-2}b)}.$$

So we have $F_N^{Lie,(2)} = F_N$ and $\oplus_{N \in \mathbb{N}} F_N^{Lie,(n)} = \bigoplus_{x_1, \dots, x_{n-2} \in \{a,b\}} F^{(ax_1 \dots x_{n-2}b)}$. Then $\delta_F^{(3)}$ maps $F_N^{Lie,(2)}$ to $F_{N+1}^{Lie,(3)}$ and $\delta_F^{(4)}$ maps $F_N^{Lie,(3)}$ to $F_{N+1}^{Lie,(4)}$.

Let us set

$$H_N^2 = \mathrm{Ker}(\delta_{3|F_N^{Lie,(2)}}^{(F)})$$

and

$$H_N^3 = \mathrm{Ker}(\delta_{4|F_N^{Lie,(2)}}^{(F)})/\delta_3^{(F)}(F_{N-1}^{Lie,(2)}).$$

Then if we set

$$H^2 = \mathrm{Ker}(\delta_3^{Lie,(F)}) \quad \text{and} \quad H^3 = \mathrm{Ker}(\delta_4^{Lie,(F)})/\mathrm{Im}(\delta_3^{Lie,(F)}),$$

we have $H^2 = \oplus_{N \geq 0} H_N^2$ and $H^3 = \oplus_{N \geq 0} H_N^3$.

3.4.2. *Results.*

Theorem 3.3. *1) H_N^2 is zero if $N \neq 1$, and H_1^2 is one-dimensional, spanned by the class of $r = x_1 \otimes y_1$.*
2) H_N^3 is zero if $N \neq 2$, and H_2^3 is two-dimensional, spanned by the classes of $[r^{(13)}, r^{(23)}] \in F_2^{(aab)}$ and $[r^{(12)}, r^{(13)}] \in F_2^{(abb)}$.

Proof. See Appendix D. $\qquad\square$

3.5. **Solution of the universal Lie QYB equations.**

Theorem 3.4. *There exists a unique solution $(\varrho_n)_{n \geq 1} \in \prod_{n \geq 1} F_n$ to the universal Lie QYB equations (16), such that $\varrho_1 = x_1 \otimes y_1$.*

Proof. ϱ_2 should satisfy $\varrho_2 \in F_2$ and

$$\delta_3^{(F)}(\varrho_2) = -2\mu_{Lie}([w_1, v_1] \otimes [w_1', u_1] \otimes [v_1', u_1']). \tag{26}$$

Denote by ς the right side of (26). Then ς belongs to $F_3^{Lie,(3)}$, and it follows from Theorem 3.2 that $\delta_4^{(F)}(\varsigma) = 0$. Then the second part of Theorem 3.3 implies the existence of ϱ_2 satisfying (26), and the first part of this Theorem implies the unicity of ϱ_2. The sequence $(\varrho_n)_{n \geq 3}$ is then constructed inductively in the same way. $\square$

4. Quantization of Lie bialgebras

In this Section, we again fix an element ω of $\mathcal{B}(\mathbb{K}[[\hbar]])$.

4.1. Lie bialgebras.

Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra (we may also assume that $\mathfrak{g}$ is a positively graded Lie bialgebra, whose homogeneous components are finite-dimensional). We denote by $[\,,\,]_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and by $\delta_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ the bracket and cobracket of $\mathfrak{g}$. We also denote by $[\,,\,]_{\mathfrak{g}^*}$ and $\delta_{\mathfrak{g}^*}$ the bracket and cobracket of $\mathfrak{g}^*$, so $[\,,\,]_{\mathfrak{g}^*} = (\delta_{\mathfrak{g}})^*$ and $\delta_{\mathfrak{g}^*} = ([\,,\,]_{\mathfrak{g}})^*$. We denote by $\mathfrak{d}$ the double Lie bialgebra of $\mathfrak{g}$, by $[\,,\,]_{\mathfrak{d}}$ and $\delta_{\mathfrak{d}}$ its bracket and cobracket. We have $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, $([\,,\,]_{\mathfrak{d}})_{|\mathfrak{g} \times \mathfrak{g}} = [\,,\,]_{\mathfrak{g}}$, $([\,,\,]_{\mathfrak{d}})_{|\mathfrak{g}^* \times \mathfrak{g}^*} = [\,,\,]_{\mathfrak{g}^*}$ $(\delta_{\mathfrak{d}})_{|\mathfrak{g}} = \delta_{\mathfrak{g}}$ and $(\delta_{\mathfrak{d}})_{|\mathfrak{g}^*} = -\delta_{\mathfrak{g}^*}$.

Moreover, the symmetric nondegenerate bilinear form $\langle\,,\,\rangle_{\mathfrak{d}}$ defined on $\mathfrak{d}$ by $\langle (x, \xi), (y, \eta) \rangle_{\mathfrak{d}} = \eta(x) + \xi(y)$ is nondegenerate and invariant, and if $r_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^*$ is the canonical element of $\mathfrak{g} \otimes \mathfrak{g}^*$ corresponding to the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, we have $\delta_{\mathfrak{d}}(x) = [x \otimes 1 + 1 \otimes x, r_{\mathfrak{g}}]_{\mathfrak{d}}$.

4.2. Subalgebras of shuffle algebras.

Recall that $\mathrm{Sh}^{\omega}(\mathfrak{d})$ is a topological $\mathbb{K}[[\hbar]]$-Hopf algebra, linearly isomorphic to $T(\mathfrak{d})[[\hbar]] = \oplus_{k \geq 0} \mathfrak{d}^{\otimes k}[[\hbar]]$. Then its subspaces $T(\mathfrak{g})$ and $T(\mathfrak{g}^*)$ are Hopf subalgebras, isomorphic to $\mathrm{Sh}^{\omega}(\mathfrak{g})$ and $\mathrm{Sh}^{\omega}(\mathfrak{g}^*)$.

Let us define $\mathrm{Sh}^{\omega}_{\hbar}(\mathfrak{g})$ (resp., $\mathrm{Sh}^{\omega}_{\hbar}(\mathfrak{g}^*)$) as the subspace $\widehat{\bigoplus}_{k \geq 0} \hbar^k \mathfrak{g}^{\otimes k}[[\hbar]]$ (resp., $\widehat{\bigoplus}_{k \geq 0} \hbar^k (\mathfrak{g}^*)^{\otimes k}[[\hbar]]$) of $\mathrm{Sh}^{\omega}(\mathfrak{d})$, where $\widehat{\oplus}$ means the complete direct sum. Then $\mathrm{Sh}^{\omega}_{\hbar}(\mathfrak{g})$ (resp., $\mathrm{Sh}^{\omega}_{\hbar}(\mathfrak{g}^*)$) is a topological Hopf subalgebra of $\mathrm{Sh}^{\omega}(\mathfrak{g})$ (resp., of $\mathrm{Sh}^{\omega}(\mathfrak{g}^*)$).

Remark 6. If we emphasize the dependence of the shuffle algebra $\mathrm{Sh}^{\omega}(\mathfrak{a})$ of a Lie algebra $(\mathfrak{a}, [\,,\,]_{\mathfrak{a}})$ in the Lie bracket of $\mathfrak{a}$ by denoting it $\mathrm{Sh}(\mathfrak{a}, [\,,\,]_{\mathfrak{a}})$, then $\mathrm{Sh}^{\omega}_{\hbar}(\mathfrak{g})$ may be viewed as a completion of $\mathrm{Sh}^{\omega}(\mathfrak{g}[[\hbar]], \hbar[\,,\,]_{\mathfrak{g}})$.

4.3. Tensor algebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and Hopf pairing.

Recall that the Hopf algebra $T^{\omega}_{\hbar}(\mathfrak{g})$ is the vector space $T(\mathfrak{g})[[\hbar]]$, equipped with the undeformed multiplication $m_{T(\mathfrak{g})}$ the tensor algebra $T(\mathfrak{g})$ and the comultiplication defined by "reversing the arrows" in the definition of the shuffle algebra $\mathrm{Sh}^{\omega}(\mathfrak{a})$ (Proposition 1.3).

There is a unique bilinear map

$$\langle\,,\,\rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} : \mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g} \to \mathbb{K}[[\hbar]][\hbar^{-1}],$$

such that if $\xi_1, \ldots, \xi_m$ belong to $\mathfrak{g}^*$ and $x_1, \ldots, x_n$ belong to $\mathfrak{g}$, then $\langle (\xi_1 \cdots \xi_m), x_1 \otimes \cdots \otimes x_n \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} = \hbar^{-n} \delta_{n,m} \prod_{i=1}^{n} \langle \xi_i, x_i \rangle_{\mathfrak{g}^* \times \mathfrak{g}}$, and $\langle\,,\,\rangle_{\mathfrak{g}^* \times \mathfrak{g}}$ is the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Then $\langle\,,\,\rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}}$ is a Hopf pairing, which means that we have

$$\langle \xi\eta, x \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} = \sum_i \langle \xi, x^{(1)} \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} \langle \eta, x^{(2)} \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} \tag{27}$$

and

$$\langle \xi, xy \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} = \sum_i \langle \xi^{(1)}, x \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} \langle \xi^{(2)}, x \rangle_{\mathrm{Sh}^{\omega}(\mathfrak{g}^*) \times T^{\omega}_{\hbar}\mathfrak{g}} \tag{28}$$

for any ξ, η in $\mathrm{Sh}^\omega(\mathfrak{g}^*)$ and any x, y in $T_\hbar^\omega(\mathfrak{g})$, where we set $\Delta_{T_\hbar^\omega(\mathfrak{g})}(x) = \sum x^{(1)} \otimes x^{(2)}$ and $\Delta_{\mathrm{Sh}^\omega(\mathfrak{g}^*)}(\xi) = \sum \xi^{(1)} \otimes \xi^{(2)}$.

4.4. R-matrix and associated Hopf algebra morphism.

Recall that $r_\mathfrak{g}$ is an element of $\mathfrak{g} \otimes \mathfrak{g}^* \subset \mathfrak{d} \otimes \mathfrak{d}$ and let us set

$$\mathcal{R} = \mathcal{R}^\omega(\sum_{n \geq 0} \rho_n^\omega(\hbar r_\mathfrak{g})).$$

Then $\mathcal{R}$ belongs to $\mathrm{Sh}^\omega(\mathfrak{g}) \widehat{\otimes} \mathrm{Sh}^\omega(\mathfrak{g}^*)$, which is a subalgebra of $\mathrm{Sh}^\omega(\mathfrak{d}) \widehat{\otimes} \mathrm{Sh}^\omega(\mathfrak{d})$. It follows from Proposition 2.1 and Remark 3 that $\mathcal{R}$ satisfies the quasitriangularity identities

$$(\Delta_{\mathrm{Sh}^\omega(\mathfrak{g})} \otimes \mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{g}^*)})(\mathcal{R}) = \mathcal{R}^{(13)}\mathcal{R}^{(23)}, \quad (\mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{g})} \otimes \Delta_{\mathrm{Sh}^\omega(\mathfrak{g}^*)})(\mathcal{R}) = \mathcal{R}^{(13)}\mathcal{R}^{(12)} \tag{29}$$

and

$$(S_{\mathrm{Sh}^\omega(\mathfrak{g})} \otimes \mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{g}^*)})(\mathcal{R}) = (\mathrm{id}_{\mathrm{Sh}^\omega(\mathfrak{g})} \otimes S^{-1}_{\mathrm{Sh}^\omega(\mathfrak{g}^*)})(\mathcal{R}), \tag{30}$$

and from Theorem 0.1 that it satisfies the QYBE

$$\mathcal{R}^{(12)}\mathcal{R}^{(13)}\mathcal{R}^{(23)} = \mathcal{R}^{(23)}\mathcal{R}^{(13)}\mathcal{R}^{(12)}. \tag{31}$$

Lemma 4.1. *The rule*

$$\ell(x) = \langle \mathcal{R}, \mathrm{id} \otimes x \rangle_{\mathrm{Sh}^\omega(\mathfrak{g}^*) \times T_\hbar^\omega(\mathfrak{g})}$$

for $x \in T_\hbar^\omega(\mathfrak{g})$ defines a linear map ℓ from $T_\hbar^\omega(\mathfrak{g})$ to $\mathrm{Sh}^\omega(\mathfrak{g})$.

Proof. Let us write $\mathcal{R} = \sum_{n \geq 0} \hbar^n \mathcal{R}_n$, then $\mathcal{R}_n$ has the following form

$$\mathcal{R}_n \in \sum_{i_1, \dots, i_n \in I, \sigma \in \mathfrak{S}_n} (a_{i_1} \cdots a_{i_n}) \otimes (b_{i_{\sigma(1)}} \cdots b_{i_{\sigma(n)}}) + \bigoplus_{(k,k') | k \leq n, k' \leq n, (k,k') \neq (n,n)} \mathfrak{g}^{\otimes k} \otimes (\mathfrak{g}^*)^{\otimes k},$$

where $(a_i)_{i \in I}$ is a basis of $\mathfrak{g}$ and $(b_i)_{i \in I}$ is the dual basis of $\mathfrak{g}^*$. Then if $x_1, \dots, x_n$ are in $\mathfrak{g}$ and $x = x_1 \otimes \cdots \otimes x_k$, $\langle \mathcal{R}_k, \mathrm{id} \otimes x \rangle_{\mathrm{Sh}^\omega(\mathfrak{g}^*) \times T_\hbar^\omega(\mathfrak{g})} = 0$ if $k < n$, so $\ell(x)$ has nonnegative $\hbar$-adic valuation. More precisely, we have

$$\mathcal{R}_n \in \sum_{i_1, \dots, i_n \in I} (a_{i_1}) \cdots (a_{i_n}) \otimes (b_{i_1} \cdots b_{i_n}) + (\bigoplus_{k | k \leq n} \mathfrak{g}^{\otimes k}) \otimes (\bigoplus_{k' | k' < n} (\mathfrak{g}^*)^{\otimes k'}),$$

so $\langle \mathcal{R}_n, \mathrm{id} \otimes x \rangle_{\mathrm{Sh}^\omega(\mathfrak{g}^*) \times T_\hbar^\omega(\mathfrak{g})} = \hbar^{-n}(x_1) \cdots (x_n)$. Therefore

$$\ell(x_1 \otimes \cdots \otimes x_k) = (x_1) \cdots (x_k) + O(\hbar). \tag{32}$$

$\square$

Then it follows from equations (29), (30), (27) and (28) that ℓ is a morphism of Hopf algebras from $T_\hbar^\omega(\mathfrak{g})^{opp}$ to $\mathrm{Sh}^\omega(\mathfrak{g})$ ($T_\hbar^\omega(\mathfrak{g})^{opp}$ is the opposite algebra of $T_\hbar^\omega(\mathfrak{g})$, endowed with the same coproduct as $T_\hbar^\omega(\mathfrak{g})$).

4.5. **Construction of $U_\hbar^\omega \mathfrak{g}$.** Let us set $U_\hbar^\omega \mathfrak{g} = \mathrm{Im}(\ell)$. Since ℓ is a morphism of Hopf algebras, $U_\hbar^\omega \mathfrak{g}$ is a Hopf subalgebra of $\mathrm{Sh}^\omega(\mathfrak{g})$. We will denote by $m_{U_\hbar^\omega \mathfrak{g}}$ and $\Delta_{U_\hbar^\omega \mathfrak{g}}$ the product and coproduct of $U_\hbar^\omega \mathfrak{g}$, and by $\varepsilon_{U_\hbar^\omega \mathfrak{g}}$ and $S_{U_\hbar^\omega \mathfrak{g}}$ its counit and antipode.

Theorem 4.1. $(U_\hbar^\omega \mathfrak{g}, m_{U_\hbar^\omega \mathfrak{g}}, \Delta_{U_\hbar^\omega \mathfrak{g}}, \varepsilon_{U_\hbar^\omega \mathfrak{g}}, S_{U_\hbar \mathfrak{g}})$ *is a quantization of* $(\mathfrak{g}, [\ , \ ]_\mathfrak{g}, \delta_\mathfrak{g})$.
   Moreover, $\mathrm{Im}(\ell)$ *is a divisible submodule of* $\mathrm{Sh}^\omega(\mathfrak{g})$, *i.e.* $\mathrm{Im}(\ell) \cap \hbar \, \mathrm{Sh}^\omega(\mathfrak{g}) = \hbar \, \mathrm{Im}(\ell)$.

   Proof. Let us show that $U_\hbar^\omega \mathfrak{g} / \hbar U_\hbar^\omega \mathfrak{g}$ is isomorphic to $U\mathfrak{g}$. $U_\hbar^\omega \mathfrak{g}$ is isomorphic, as a Hopf algebra, to $T_\hbar^\omega(\mathfrak{g})^{opp} / \mathrm{Ker}(\ell)$, therefore $U_\hbar^\omega \mathfrak{g} / \hbar U_\hbar^\omega \mathfrak{g}$ is isomorphic to the Hopf algebra $T_\hbar^\omega(\mathfrak{g})^{opp} / \big(\mathrm{Ker}(\ell) + \hbar T_\hbar^\omega(\mathfrak{g})^{opp} \big)$. To identify the latter Hopf algebra with $U\mathfrak{g}$, let us study the kernel $\mathrm{Ker}(\ell)$.

Lemma 4.2. *For any integers p and q, there exists unique $(q+1)$-linear maps β_{pq} and $\gamma_{qp} : \mathfrak{g}^{q+1} \to \mathfrak{g}^{\otimes p}[[\hbar]]$, such that for any $\xi_1, \ldots, \xi_p$ in $\mathfrak{g}^*$ and any $x, x_1, \ldots, x_q$ in $\mathfrak{g}$, we have the equalities*

$$\langle B_{pq}(\xi_1, \ldots, \xi_p | x_1, \ldots, x_q), x \rangle_\mathfrak{d} = \langle \xi_1 \otimes \cdots \otimes \xi_p, \beta_{pq}(x_1, \ldots, x_q, x) \rangle_{\mathfrak{d}^{\otimes p}}$$

and

$$\langle B_{qp}(x_1, \ldots, x_q | \xi_1, \ldots, \xi_p), x \rangle_\mathfrak{d} = \langle \xi_1 \otimes \cdots \otimes \xi_p, \gamma_{qp}(x_1, \ldots, x_q, x) \rangle_{\mathfrak{d}^{\otimes p}}$$

(we view $\mathfrak{g}$ and $\mathfrak{g}^*$ as subspaces of $\mathfrak{d}$ via the maps $x \mapsto (x, 0)$ and $\xi \mapsto (0, \xi)$, and $\langle \ , \ \rangle_{\mathfrak{d}^{\otimes p}}$ is the pth tensor power of $\langle \ , \ \rangle_\mathfrak{d}$).

   Extend the pairing $\langle \ , \ \rangle_{\mathrm{Sh}^\omega(\mathfrak{g}^*) \times T_\hbar^\omega(\mathfrak{g})}$ to a bilinear map

$$\langle \ , \ \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} : \mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g}) \to \mathbb{K}[[\hbar]][\hbar^{-1}],$$

by the rule that if $x_1, \ldots, x_n$ are elements of $\mathfrak{d}$, one of which belongs to $\mathfrak{g}$, and y is any element of $T_\hbar^\omega(\mathfrak{g})$, then $\langle (x_1 \cdots x_n), y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d})[[\hbar]] \times T_\hbar^\omega(\mathfrak{g})} = 0$.

Lemma 4.3. *There exist unique bilinear maps ϕ and $\psi : \mathrm{Sh}^\omega(\mathfrak{g}) \times T_\hbar^\omega(\mathfrak{g}) \to T_\hbar^\omega(\mathfrak{g})$ such that for any $(x, x', y) \in \mathrm{Sh}^\omega(\mathfrak{g}) \times \mathrm{Sh}^\omega(\mathfrak{g}^*) \times T_\hbar^\omega(\mathfrak{g})$, we have*

$$\langle x'x, y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} = \langle x', \phi(x, y) \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})}$$

and

$$\langle xx', y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} = \langle x', \psi(x, y) \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})}.$$

Identify $\mathfrak{g}$ with a subspace of $T_\hbar^\omega(\mathfrak{g})$ of tensors of degree 1. Then if x and y belong to $\mathfrak{g}$, then

$$\phi(x, y) \in \frac{1}{2}[x, y]_\mathfrak{g} + \hbar T_\hbar^\omega(\mathfrak{g}), \quad \psi(x, y) \in -\frac{1}{2}[x, y]_\mathfrak{g} + \hbar T_\hbar^\omega(\mathfrak{g}).$$

Proof. If n and m are integers ≥ 0, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are elements of $\mathfrak{g}$, and if $x = (x_1 \cdots x_n)$ and $y = y_1 \otimes \cdots \otimes y_m$, we set

$$\phi(x, y) = \sum_{(\lambda_1, \ldots, \lambda_m) \text{ partition of } n} \sum_{k_1, \ldots, k_m > 0}$$

$$\hbar^{k_1 + \cdots + k_m - m} \beta_{k_1 \lambda_1}(x_1, \ldots, x_{\lambda_1}, y_1) \otimes \cdots \otimes \beta_{k_m \lambda_m}(x_{\lambda_1 + \cdots + \lambda_{m-1} + 1}, \ldots, x_{\lambda_1 + \cdots + \lambda_m}, y_m)$$

and

$$\psi(x, y) = \sum_{(\lambda_1, \ldots, \lambda_m) \text{ partition of } n} \sum_{k_1, \ldots, k_m > 0}$$

$$\hbar^{k_1 + \cdots + k_m - m} \gamma_{\lambda_1 k_1}(x_1, \ldots, x_{\lambda_1}, y_1) \otimes \cdots \otimes \gamma_{\lambda_m k_m}(x_{\lambda_1 + \cdots + \lambda_{m-1} + 1}, \ldots, x_{\lambda_1 + \cdots + \lambda_m}, y_m).$$

In both right hand sides, only nonnegative powers of $\hbar$ occur, and the coefficient of each power of $\hbar$ is a finite sum, so that both right sides belong to $T_\hbar^\omega(\mathfrak{g})$. It is easy to check that these are the unique maps satisfying the above requirements. The last statements follow from the equalities $\beta_{11}(x, y) = \frac{1}{2}[x, y]_\mathfrak{g}$ and $\gamma_{11}(x, y) = -\frac{1}{2}[x, y]_\mathfrak{g}$. Both equalities follow from the invariance of $\langle \, , \, \rangle_\mathfrak{d}$. $\square$

Lemma 4.4. *If x and y belong to $T_\hbar^\omega(\mathfrak{g})$, and we write $\Delta_{T_\hbar^\omega(\mathfrak{g})}(y) = \sum y^{(1)} \otimes y^{(2)}$, then*

$$\sum y^{(1)} \phi(\ell(y^{(2)}), x) - \sum \psi(\ell(y^{(1)}), x) y^{(2)}$$

belongs to $\mathrm{Ker}(\ell)$.

Proof. We have

$$\langle \mathcal{R}^{(12)} \mathcal{R}^{(13)} \mathcal{R}^{(23)}, \mathrm{id} \otimes x \otimes y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})}$$

$$= \sum \langle \mathcal{R}^{(12)} \mathcal{R}^{(13)} \mathcal{R}^{(24)}, \mathrm{id} \otimes x \otimes y^{(1)} \otimes y^{(2)} \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})}$$

$$= \sum \langle \mathcal{R}^{(12)}(1 \otimes \ell(y^{(2)})), \mathrm{id} \otimes x \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} \ell(y^{(1)}).$$

Lemma 4.3 implies the identity

$$\langle \mathcal{R}(1 \otimes x), \mathrm{id} \otimes y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} = \ell(\phi(x, y)),$$

and since ℓ is an algebra antihomomorphism from $T_\hbar^\omega(\mathfrak{g})$ to $\mathrm{Sh}^\omega(\mathfrak{g})$, we get

$$\langle \mathcal{R}^{(12)} \mathcal{R}^{(13)} \mathcal{R}^{(23)}, \mathrm{id} \otimes x \otimes y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} = \ell(\sum y^{(1)} \phi(\ell(y^{(2)}), x)).$$

In the same way, $\langle \mathcal{R}^{(23)} \mathcal{R}^{(13)} \mathcal{R}^{(12)}, \mathrm{id} \otimes x \otimes y \rangle_{\mathrm{Sh}^\omega(\mathfrak{d}) \times T_\hbar^\omega(\mathfrak{g})} = \ell(\sum \psi(\ell(y^{(1)}), x) y^{(2)})$. Since $\mathcal{R}$ satisfies QYBE, we have $\ell(\sum y^{(1)} \phi(\ell(y^{(2)}), x)) = \ell(\sum \psi(\ell(y^{(1)}), x) y^{(2)})$. $\square$

End of proof of Theorem 4.1. If $x \in \mathfrak{g} \subset T_\hbar^\omega(\mathfrak{g})$, then $\Delta_{T_\hbar^\omega(\mathfrak{g})}(x) = x \otimes 1 + 1 \otimes x + O(\hbar)$. Then applying Lemma 4.4 to the case when x and y belong to $\mathfrak{g}$

and using the end of Lemma 4.3, we construct bilinear maps $m_n : \mathfrak{g} \times \mathfrak{g} \to T(\mathfrak{g})$, where $n \geq 1$, such that if x and y belong to $\mathfrak{g}$,

$$y \otimes x - x \otimes y - [x, y]_{\mathfrak{g}} - \sum_{n|n \geq 1} \hbar^n m_n(x, y) \tag{33}$$

belongs to $\mathrm{Ker}(\ell)$.

Let us denote by I_0 the complete two-sided ideal of $T_\hbar^\omega(\mathfrak{g})^{opp}$ generated by the elements (33). Then $I_0 \subset \mathrm{Ker}(\ell)$. We have therefore a surjective morphism of $\mathbb{K}[[\hbar]]$-algebras $T_\hbar^\omega(\mathfrak{g})^{opp}/I_0 \to T_\hbar^\omega(\mathfrak{g})^{opp}/\mathrm{Ker}(\ell)$. The reduction modulo $\hbar$ of this map is also surjective; it is a map $T_\hbar^\omega(\mathfrak{g})^{opp}/(I_0 + \hbar T_\hbar^\omega(\mathfrak{g})^{opp}) \to T_\hbar^\omega(\mathfrak{g})^{opp}/(\mathrm{Ker}(\ell) + \hbar T_\hbar^\omega(\mathfrak{g})^{opp})$. Due to the form of I_0, $T_\hbar^\omega(\mathfrak{g})^{opp}/(I_0 + \hbar T_\hbar^\omega(\mathfrak{g})^{opp})$ is isomorphic to $U\mathfrak{g}$. As a result, we obtain that there is a unique surjective morphism $s : U\mathfrak{g} \to T_\hbar^\omega(\mathfrak{g})^{opp}/(\mathrm{Ker}(\ell) + \hbar T_\hbar^\omega(\mathfrak{g})^{opp})$, such that for any $x \in \mathfrak{g}$, $s(x) =$ the class of x.

On the other hand, recall that $U\mathfrak{g}$ is also the subalgebra of $\mathrm{Sh}^\omega(\mathfrak{g})$ generated by the elements of degree 1. $\mathrm{Im}(\ell)$ is a subalgebra of $\mathrm{Sh}^\omega(\mathfrak{g})$. It follows from equation (32) that the image of $\mathrm{Im}(\ell)$ by the morphism $\mathrm{Sh}^\omega(\mathfrak{g}) \to \mathrm{Sh}^\omega(\mathfrak{g})$ given by the reduction modulo $\hbar$ is exactly $U\mathfrak{g}$. This means that we have an isomorphism of algebras between $U\mathfrak{g}$ and $\mathrm{Im}(\ell)/(\hbar\,\mathrm{Sh}^\omega(\mathfrak{g}) \cap \mathrm{Im}(\ell))$. The latter algebra is a quotient algebra of $\mathrm{Im}(\ell)/\hbar\,\mathrm{Im}(\ell)$. We obtain that there exists a surjective algebra morphism $s' : \mathrm{Im}(\ell)/\hbar\,\mathrm{Im}(\ell) \to U\mathfrak{g}$, such that for any $x \in \mathfrak{g}$, $s'($the class of $\ell(x)) = x$.

Moreover, ℓ induces an isomorphism between $T_\hbar^\omega(\mathfrak{g})^{opp}/\mathrm{Ker}(\ell)$ and $\mathrm{Im}(\ell)$, so its reduction modulo $\hbar$ induces an isomorphism between $T_\hbar^\omega(\mathfrak{g})^{opp}/(\mathrm{Ker}(\ell) + \hbar T_\hbar^\omega(\mathfrak{g})^{opp})$ and $\mathrm{Im}(\ell)/\hbar\,\mathrm{Im}(\ell)$. Denote by $\bar{\ell}$ this reduction, then we have $\bar{\ell} = s \circ s'$. Since s and s' is surjective, s is an isomorphism between $U\mathfrak{g}$ and $\mathrm{Im}(\ell)/\hbar\,\mathrm{Im}(\ell)$. In other words, we have shown that $U_\hbar^\omega\mathfrak{g}/\hbar U_\hbar^\omega\mathfrak{g}$ is isomorphic to $U\mathfrak{g}$.

Moreover, $U_\hbar^\omega\mathfrak{g}$ is $\hbar$-adically complete, and it is torsion-free, because it is a $\mathbb{K}[[\hbar]]$-submodule of $\mathrm{Sh}^\omega(\mathfrak{g})$. $U_\hbar^\omega\mathfrak{g}$ is therefore a topologically free $\mathbb{K}[[\hbar]]$-algebra, such that $U_\hbar^\omega\mathfrak{g}/U_\hbar^\omega\mathfrak{g} = U\mathfrak{g}$.

Let us study now the co-Poisson structure on $U\mathfrak{g}$ induced by this isomorphism. Consider $x \in \mathfrak{g}$ as an element of $T_\hbar^\omega(\mathfrak{g})^{opp}$. Then

$$\frac{1}{\hbar}(\Delta_{T_\hbar^\omega(\mathfrak{g})}(x) - \Delta'_{T_\hbar^\omega(\mathfrak{g})}(x)) \in \delta_{\mathfrak{g}}(x) + \hbar T_\hbar^\omega(\mathfrak{g})\widehat{\otimes} T_\hbar^\omega(\mathfrak{g}),$$

where $\hat{\otimes}$ is the $\hbar$-adically completed tensor product. Taking the image by ℓ of this identity, we find

$$\frac{1}{\hbar}(\Delta_{U_\hbar^\omega\mathfrak{g}}(\ell(x)) - \Delta'_{U_\hbar^\omega\mathfrak{g}}(\ell(x))) \in \ell^{\otimes 2}(\delta_{\mathfrak{g}}(x)) + \hbar U_\hbar^\omega(\mathfrak{g})\widehat{\otimes} U_\hbar^\omega(\mathfrak{g}),$$

which means that the co-Poisson structure on $U\mathfrak{g}$ corresponding to $U_\hbar^\omega\mathfrak{g}$ is given by $\delta_{\mathfrak{g}}$.

Let us now prove that $\mathrm{Im}(\ell)$ is a divisible submodule of $\mathrm{Sh}^\omega(\mathfrak{g})$. We have shown that the map s' is an isomorphism, which means that the surjective morphism

$\mathrm{Im}(\ell)/\hbar\,\mathrm{Im}(\ell) \to \mathrm{Im}(\ell)/(\mathrm{Im}(\ell) \cap \hbar\,\mathrm{Sh}^\omega(\mathfrak{g}))$ is an isomorphism. This implies that $\mathrm{Im}(\ell) \cap \hbar\,\mathrm{Sh}^\omega(\mathfrak{g}) = \hbar\,\mathrm{Im}(\ell)$. $\qquad\square$

4.6. **Functoriality.**

Proposition 4.1. *The map* $(\mathfrak{g},[\,,\,]_{\mathfrak{g}},\delta_{\mathfrak{g}}) \mapsto U_\hbar^\omega\mathfrak{g}$ *defines a universal quantization functor from the category of finite-dimensional Lie bialgebras.*

Proof. Let ϕ be a morphism of Lie bialgebras from $(\mathfrak{g},[\,,\,]_{\mathfrak{g}},\delta_{\mathfrak{g}})$ to $(\mathfrak{h},[\,,\,]_{\mathfrak{h}},\delta_{\mathfrak{h}})$. Then ϕ induces Lie algebra morphisms $\phi_{\mathfrak{g}\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ and $\phi_{\mathfrak{h}^*\mathfrak{g}^*} : \mathfrak{h}^* \to \mathfrak{g}^*$. The first morphism induces a Hopf algebra morphism $\mathrm{Sh}(\phi_{\mathfrak{g}\mathfrak{h}}) : \mathrm{Sh}^\omega(\mathfrak{g}) \to \mathrm{Sh}^\omega(\mathfrak{h})$, and the dual to $\mathrm{Sh}^\omega(\phi_{\mathfrak{h}^*\mathfrak{g}^*})$ induces a Hopf algebra morphism $T(\phi_{\mathfrak{g}\mathfrak{h}}) : T_\hbar^\omega(\mathfrak{g}) \to T_\hbar^\omega(\mathfrak{h})$. Let us denote by $\mathcal{R}_\mathfrak{g}$ and $\mathcal{R}_\mathfrak{h}$ the analogues of $\mathcal{R}$ for $\mathfrak{g}$ and $\mathfrak{h}$; then $\mathcal{R}_\mathfrak{g}$ belongs to $\mathrm{Sh}^\omega(\mathfrak{g})\widehat{\otimes}\,\mathrm{Sh}^\omega(\mathfrak{g}^*)$ and $\mathcal{R}_\mathfrak{h}$ belongs to $\mathrm{Sh}^\omega(\mathfrak{h})\widehat{\otimes}\,\mathrm{Sh}^\omega(\mathfrak{h}^*)$. Moreover, if $r_\mathfrak{g}$ and $r_\mathfrak{h}$ are the canonical r-matrices of $\mathfrak{g}$ and $\mathfrak{h}$, we have $(\mathrm{id}\otimes\phi_{\mathfrak{h}^*\mathfrak{g}^*})(r_\mathfrak{g}) = (\phi_{\mathfrak{g}\mathfrak{h}} \otimes \mathrm{id})(r_\mathfrak{h})$, therefore

$$(\mathrm{id} \otimes \mathrm{Sh}(\phi_{\mathfrak{h}^*\mathfrak{g}^*}))(\mathcal{R}_\mathfrak{h}) = (\mathrm{Sh}(\phi_{\mathfrak{g}\mathfrak{h}}) \otimes \mathrm{id})(\mathcal{R}_\mathfrak{g}).$$

Let us denote by $\ell_\mathfrak{g}$ and $\ell_\mathfrak{h}$ the analogues of ℓ corresponding to $\mathfrak{g}$ and $\mathfrak{h}$. If $x \in T_\hbar^\omega\mathfrak{g}$, we have

$$\begin{aligned}
\mathrm{Sh}(\phi_{\mathfrak{g}\mathfrak{h}})(\ell_\mathfrak{g}(x)) &= \langle\mathrm{id}\otimes x, (\mathrm{Sh}(\phi_{\mathfrak{g}\mathfrak{h}}) \otimes \mathrm{id})(\mathcal{R}_\mathfrak{g})\rangle_{T_\hbar^\omega(\mathfrak{g})\times\mathrm{Sh}^\omega(\mathfrak{g}^*)}\\
&= \langle\mathrm{id}\otimes x, (\mathrm{id}\otimes\mathrm{Sh}(\phi_{\mathfrak{h}^*\mathfrak{g}^*}))(\mathcal{R}_\mathfrak{h})\rangle_{T_\hbar^\omega(\mathfrak{g})\times\mathrm{Sh}^\omega(\mathfrak{g}^*)}\\
&= \langle\mathrm{id}\otimes T(\phi_{\mathfrak{g}\mathfrak{h}})(x), \mathcal{R}_\mathfrak{h}\rangle_{T_\hbar^\omega(\mathfrak{h})\times\mathrm{Sh}^\omega(\mathfrak{h}^*)} = \ell_\mathfrak{h}(T(\phi_{\mathfrak{g}\mathfrak{h}})(x)),
\end{aligned}$$

so $\mathrm{Sh}(\phi_{\mathfrak{g}\mathfrak{h}}) \circ \ell_\mathfrak{g} = \ell_\mathfrak{h} \circ T(\phi_{\mathfrak{g}\mathfrak{h}})$. Therefore the restriction of $\mathrm{Sh}(\phi_{\mathfrak{g}\mathfrak{h}})$ to $U_\hbar^\omega\mathfrak{g}$ induces a Hopf algebra morphism from $U_\hbar^\omega\mathfrak{g}$ to $U_\hbar^\omega\mathfrak{h}$. Let us denote by ϕ^U this morphism; it is then clear that the reduction mod $\hbar$ of ϕ^U coincides with the morphism from $U\mathfrak{g}$ to $U\mathfrak{h}$ induced by ϕ. Moreover, if $\psi : \mathfrak{h} \to \mathfrak{k}$ is a morphism of Lie balgebras, we have $\mathrm{Sh}(\psi \circ \phi) = \mathrm{Sh}(\psi) \circ \mathrm{Sh}(\psi)$; the restriction of this identity to $U_\hbar^\omega\mathfrak{g}$ yields $(\psi \circ \phi)^U = \psi^U \circ \phi^U$.

Finally, the form taken by the relations (33) shows that the quantization functor $\mathfrak{g} \mapsto U_\hbar^\omega\mathfrak{g}$ is universal. $\qquad\square$

Remark 7. The condition $p_{kl} = 0$ if $k > l$ in the definition of the spaces $F^{(x_1\dots x_n)}$ (see (19)) seems to imply that the ideal I_0 generated by elements (33) is defined in terms of acyclic tensor calculus, in the sense of [6]. $\qquad\square$

Remark 8. When $\mathfrak{g}$ is infinite-dimensional, the maps $m_P : \mathfrak{g} \to \mathfrak{g}^{\otimes n}$ defined by $m_P(x) = \sum_{i_1,\dots,i_n\in I}\langle x, P(b_{i_1},\dots,b_{i_n})\rangle_{\mathfrak{g}\times\mathfrak{g}^*}a_{i_1} \otimes \cdots \otimes a_{i_n}$, where P is a Lie polynomial, do not make sense any more. However, if $\mathfrak{g}$ is finite-dimensional, these maps are linear combinations of the maps $T_\sigma \circ (\delta\otimes\mathrm{id}^{\otimes(n-1)}) \circ \cdots (\delta\otimes\mathrm{id}) \circ \delta$, where $\sigma \in \mathfrak{S}_n$ and $\sigma \mapsto T_\sigma$ is the action of the group $\mathfrak{S}_n$ by permutation of the factors of $\mathfrak{g}^{\otimes n}$. It is easy to see that the map ℓ only involves linear combinations of compositions of the maps m_P and of the Lie bracket of $\mathfrak{g}$. It has therefore a

natural analogue when $\mathfrak{g}$ is an infinite-dimensional Lie bialgebra. It is natural to expect that the corresponding analogue of $\mathfrak{g} \mapsto U_\hbar^\omega \mathfrak{g}$ defines a quantization functor for the category of (possibly infinite-dimensional) Lie bialgebras. $\square$

4.7. **The QFSH algebra of $U_\hbar^\omega \mathfrak{g}$.** To any quantized universal enveloping algebra $U_\hbar \mathfrak{a}$, one associates in a canonical way a quantized formal series Hopf (QFSH) algebra $\mathcal{O}_\hbar(A^*)$ (see [5, 10]). If $(\mathfrak{a}, [\ ,\ ]_\mathfrak{a}, \delta_\mathfrak{a})$ is the Lie bialgebra corresponding to the semiclassical limit of $U_\hbar \mathfrak{a}$, then $\mathcal{O}_\hbar(A^*)$ is a quantization of the formal series Hopf algebra of functions on the formal group A^* corresponding to the Lie algebra $(\mathfrak{a}^*, \delta_\mathfrak{a}^*)$, endowed with the Poisson-Lie structure corresponding to the cobracket $[\ ,\ ]_\mathfrak{a}^*$.

(For example, if $\delta_\mathfrak{a} = 0$, then A^* is the additive group $\mathfrak{a}^*$, endowed with the Kostant-Kirillov Poisson bracket. If moreover $U_\hbar \mathfrak{a}$ is $U\mathfrak{a}[[\hbar]]$, then $\mathcal{O}_\hbar(A^*)$ is a deformation quantization of the formal series ring $S[[\mathfrak{a}]]$ corresponding to the Kostant-Kirillov bracket, endowed with the cocommutative coproduct such that the elements of $\mathfrak{a}$ are primitive.)

The purpose of this Section is to express the QFSH algebra of $U_\hbar^\omega \mathfrak{g}$ in terms of the above construction.

Recall that we defined $\mathrm{Sh}_\hbar^\omega(\mathfrak{g})$ as the subalgebra $\widehat{\bigoplus}_{k \geq 0} \hbar^k \mathfrak{g}^{\otimes k}[[\hbar]]$ of $\mathrm{Sh}^\omega(\mathfrak{g})$.

Proposition 4.2. *Let us define $\mathcal{O}_\hbar^\omega(G^*)$ as the intersection $\mathrm{Im}(\ell) \cap \mathrm{Sh}_\hbar^\omega(\mathfrak{g})$. Then $\mathcal{O}_\hbar^\omega(G^*)$ is the QFSH algebra of $U_\hbar^\omega \mathfrak{g}$.*

Proof. Let us first recall how the QFSH algebra of $U_\hbar^\omega \mathfrak{g}$ is defined. According to [5, 10], one defines a functor $H \mapsto H'$ in the category of topologically free Hopf algebras over $\mathbb{K}[[\hbar]]$. If H is such an algebra, let us denote by Δ_H its coproduct, by $\Delta_H^{(n)}$ its nth fold coproduct and by ε_H its counit. Let us set $\delta_n^H = (\mathrm{id}_H - \varepsilon_H)^{\otimes n} \circ \Delta_H^{(n)}$. Then H' is defined as $\{h \in H | \forall n \geq 0, \delta_n^H(h) \in \hbar^n H^{\widehat{\otimes} n}\}$. One shows ([10]) that H' is then a QFSH algebra. We first show

Lemma 4.5. $(\mathrm{Sh}^\omega(\mathfrak{g}))' = \mathrm{Sh}_\hbar^\omega(\mathfrak{g})$.

Proof. Let k and n be integers. Recall that $\mathrm{Sh}^\omega(\mathfrak{g})$ is identified, as a vector space, with $T(\mathfrak{g})[[\hbar]]$ and we denote by conc the concatenation product on $T(\mathfrak{g})$. Denote by $\mathrm{conc}^{(n)}$ the $n$fold concatenation product; $\mathrm{conc}^{(n)}$ is a linear map from $T(\mathfrak{g})^{\otimes n}$ to $T(\mathfrak{g})$. Then if $x \in \mathfrak{g}^{\otimes k} \subset \mathrm{Sh}^\omega(\mathfrak{g})$, we have

$$\mathrm{conc}^{(n)} \circ \delta_n^{\mathrm{Sh}^\omega(\mathfrak{g})}(x) = s(n, k)x,$$

where $s(n, k)$ is the number of ordered surjections from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$; so $s(n, k) = 0$ if $n < k$ and $s(n, k) > 0$ else.

Let x be an element of $\mathrm{Sh}^\omega(\mathfrak{g})$. Set $x = \sum_{k \geq 0} x_k$, where $x_k \in \mathfrak{g}^{\otimes k}[[\hbar]]$. Then if $x \in (\mathrm{Sh}^\omega(\mathfrak{g}))'$, and ν is any integer, $\delta_\nu^{\mathrm{Sh}^\omega(\mathfrak{g})}(x) \in \hbar^\nu \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes}\nu}$. Applying $\mathrm{conc}^{(\nu)}$ to this inclusion, we find $\sum_{n \geq 0} s(n, k)x_k \in \hbar^\nu \mathrm{Sh}^\omega(\mathfrak{g})$, therefore $x_k \in \hbar^\nu \mathfrak{g}^{\otimes k}[[\hbar]]$ if $k \geq \nu$. So for each k, $x_k \in \hbar^k \mathfrak{g}^{\otimes k}[[\hbar]]$, which means that $x \in \mathrm{Sh}_\hbar^\omega(\mathfrak{g})$.

Let us now show that $\mathrm{Sh}^\omega_\hbar(\mathfrak{g}) \subset (\mathrm{Sh}^\omega(\mathfrak{g}))'$. Let x belong to $\hbar^k \mathfrak{g}^{\otimes k}[[\hbar]]$, then we have, if $n \leq k$ $\delta^{\mathrm{Sh}^\omega(\mathfrak{g})}_n(x) \subset \hbar^k \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} n} \subset \hbar^n \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} n}$, and if $n > k$, $\delta^{\mathrm{Sh}^\omega(\mathfrak{g})}_n(x) = 0$ so that $\delta^{\mathrm{Sh}^\omega(\mathfrak{g})}_n(x)$ is again contained in $\hbar^n \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} n}$. So $x \in (\mathrm{Sh}^\omega(\mathfrak{g}))'$. $\square$

End of proof of Proposition 4.2. We should prove that $(\mathrm{Im}(\ell))' = \mathrm{Im}(\ell) \cap \mathrm{Sh}^\omega_\hbar(\mathfrak{g})$. By definition, $(\mathrm{Im}(\ell))' = \{x \in \mathrm{Im}(\ell) | \forall n \geq 0, \delta^{\mathrm{Im}(\ell)}_n(x) \in \hbar^n \mathrm{Im}(\ell)^{\widehat{\otimes} n}\}$.

It follows from the fact that $\mathrm{Im}(\ell)$ is a divisible submodule of $\mathrm{Sh}^\omega(\mathfrak{g})$ (see Theorem 4.1) that $\mathrm{Im}(\ell)^{\widehat{\otimes} n} \cap \hbar^n \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} n} = \hbar^n \mathrm{Im}(\ell)^{\widehat{\otimes} n}$. Therefore $(\mathrm{Im}(\ell))'$ is the same as $\{x \in \mathrm{Im}(\ell) | \forall n \geq 0, \delta^{\mathrm{Sh}^\omega(\mathfrak{g})}_n(x) \in \mathrm{Im}(\ell)^{\widehat{\otimes} n} \cap \hbar^n \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} n}\}$. Since $\delta^{\mathrm{Sh}^\omega(\mathfrak{g})}_n(\mathrm{Im}(\ell)) \subset \mathrm{Im}(\ell)^{\widehat{\otimes} n}$, this set is the same as $\{x \in \mathrm{Im}(\ell) | \forall n \geq 0, \delta^{\mathrm{Sh}^\omega(\mathfrak{g})}_n(x) \in \hbar^n \mathrm{Sh}^\omega(\mathfrak{g})^{\widehat{\otimes} n}\}$, which is $\mathrm{Im}(\ell) \cap (\mathrm{Sh}^\omega(\mathfrak{g}))'$. It then follows from Lemma 4.5 that this is $\mathrm{Im}(\ell) \cap \mathrm{Sh}^\omega_\hbar(\mathfrak{g})$. $\square$

The dual $(U^\omega_\hbar \mathfrak{g})^* = \mathrm{Hom}_{\mathbb{K}[[\hbar]]}(U^\omega_\hbar \mathfrak{g}, \mathbb{K}[[\hbar]])$ is also a QFSH algebra associated to the formal group G. If we denote by $\ell_{\mathfrak{g}^*}$ the analogue of ℓ for the Lie bialgebra $\mathfrak{g}^*$, we get therefore two quantization functors from the category of Lie bialgebra to that of QFSH algebras, namely $\mathfrak{g} \mapsto (U^\omega_\hbar \mathfrak{g})^*$ and $\mathfrak{g} \mapsto \mathcal{O}^\omega_\hbar(G) = \mathrm{Im}(\ell_{\mathfrak{g}^*}) \cap \mathrm{Sh}^\omega_\hbar(\mathfrak{g})$. Let us denote by $A \to A^\vee$ the functor associating to each Hopf algebra, its enveloping QUE (quantized universal enveloping) algebra.

Proposition 4.3. *The QFSH algebras $(U^\omega_\hbar \mathfrak{g})^*$ and $\mathcal{O}^\omega_\hbar(G) = \mathrm{Im}(\ell_{\mathfrak{g}^*}) \cap \mathrm{Sh}^\omega_\hbar(\mathfrak{g})$ are canonically isomorphic. The QUE algebras $U^\omega_\hbar \mathfrak{g}^*$ and $((U^\omega_\hbar \mathfrak{g})^*)^\vee$ are also canonically isomorphic.*

Proof. We must construct a Hopf pairing between $U^\omega_\hbar \mathfrak{g}$ and $\mathcal{O}^\omega_\hbar(G)$. The Hopf pairing $T^\omega_\hbar(\mathfrak{g}) \times \mathrm{Sh}^\omega(\mathfrak{g}^*) \to \mathbb{K}[[\hbar]][\hbar^{-1}]$ induces a pairing $T^\omega_\hbar(\mathfrak{g}) \times \mathrm{Sh}^\omega_\hbar(\mathfrak{g}^*) \to \mathbb{K}[[\hbar]]$. This pairing restricts to a pairing $T^\omega_\hbar(\mathfrak{g}) \times \big(\mathrm{Im}(\ell_{\mathfrak{g}^*}) \cap \mathrm{Sh}^\omega_\hbar(\mathfrak{g}^*)\big) \to \mathbb{K}[[\hbar]]$; now for any $x \in T^\omega_\hbar(\mathfrak{g})$ and $y \in T^\omega_\hbar(\mathfrak{g}^*)$,

$$\langle \ell_{\mathfrak{g}}(x), y \rangle_{\mathrm{Sh}^\omega(\mathfrak{g}) \times T^\omega_\hbar(\mathfrak{g}^*)} = \langle y \otimes x, \mathcal{R} \rangle_{(T^\omega_\hbar(\mathfrak{g}^*) \times \mathrm{Sh}^\omega(\mathfrak{g})) \otimes (T^\omega_\hbar(\mathfrak{g}) \times \mathrm{Sh}^\omega(\mathfrak{g}^*))} = \langle \ell_{\mathfrak{g}^*}(y), x \rangle_{\mathrm{Sh}^\omega(\mathfrak{g}^*) \times T^\omega_\hbar(\mathfrak{g})},$$

so the latter pairing descends to a Hopf pairing

$$\big(T^\omega_\hbar(\mathfrak{g}) / \mathrm{Ker}(\ell_{\mathfrak{g}})\big) \times \big(\mathrm{Im}(\ell_{\mathfrak{g}^*}) \cap \mathrm{Sh}^\omega_\hbar(\mathfrak{g}^*)\big) \to \mathbb{K}[[\hbar]],$$

which is the desired pairing $U^\omega_\hbar \mathfrak{g} \times \mathcal{O}^\omega_\hbar(G) \to \mathbb{K}[[\hbar]]$. The second part of the Proposition follows from the results of [10]. $\square$

4.8. **Behaviour of $\mathfrak{g} \mapsto U^\omega_\hbar(\mathfrak{g})$ for the double operation.** If $\mathfrak{g}$ is a Lie bialgebra, let us denote by $D(\mathfrak{g})$ is double bialgebra. If U is a QUE algebra, its quantum double $D(U)$ is the $\mathbb{K}[[\hbar]]$-module $U \otimes (U^*)^\vee$. It is a quasitriangular QUE algebra.

Proposition 4.4. *If $\mathfrak{g}$ is a Lie bialgebra, then there is an isomorphism $\iota_{\mathfrak{g}} : U^\omega_\hbar(D(\mathfrak{g})) \to D(U^\omega_\hbar(\mathfrak{g}))$. Let $\mathcal{R}_{\mathfrak{g},can}$ be the canonical R-matrix of $D(U^\omega_\hbar(\mathfrak{g}))$. Then $(\iota_{\mathfrak{g}} \otimes \iota_{\mathfrak{g}})^{-1}(\mathcal{R}_{\mathfrak{g},can})$ belongs to $(i_{\mathfrak{g},D(\mathfrak{g})})^U(U^\omega_\hbar \mathfrak{g}) \widehat{\otimes} (i_{\mathfrak{g}^*,D(\mathfrak{g})})^U(U^\omega_\hbar(\mathfrak{g}^*))$ ($i_{\mathfrak{g},D(\mathfrak{g})}$ and $i_{\mathfrak{g}^*,D(\mathfrak{g})}$ are the inclusions of $\mathfrak{g}$ and $\mathfrak{g}^*$ in $D(\mathfrak{g})$). We have $(\iota_{\mathfrak{g}^*} \otimes \iota_{\mathfrak{g}^*})^{-1}(\mathcal{R}_{\mathfrak{g}^*,can}) =$*

$(\iota_{\mathfrak{g}} \otimes \iota_{\mathfrak{g}})^{-1}(\mathcal{R}_{\mathfrak{g},can})^{(21)}$, and if $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism, then we have $(\phi^U \otimes \mathrm{id})(\iota_{\mathfrak{g}} \otimes \iota_{\mathfrak{g}})^{-1}(\mathcal{R}_{\mathfrak{g},can}) = (\mathrm{id} \otimes (\phi^*)^U)(\iota_{\mathfrak{h}} \otimes \iota_{\mathfrak{h}})^{-1}(\mathcal{R}_{\mathfrak{h},can})$.

Proof. The maps $(i_{\mathfrak{g},D(\mathfrak{g})})^U$ and $(i_{\mathfrak{g}^*,D(\mathfrak{g})})^U$ are flat deformations of the inclusions $U\mathfrak{g} \to U(D(\mathfrak{g}))$ and $U\mathfrak{g}^* \to U(D(\mathfrak{g}))$, so the composition of their tensor product with the multiplication map defines a linear isomorphism from $U_{\hbar}^{\omega}\mathfrak{g} \widehat{\otimes} U_{\hbar}^{\omega}\mathfrak{g}^*$ to $U_{\hbar}^{\omega} D(\mathfrak{g})$. Moreover, $(i_{\mathfrak{g},D(\mathfrak{g})})^U$ and $(i_{\mathfrak{g}^*,D(\mathfrak{g})})^U$ are also Hopf algebra morphisms.

Recall that we have defined a solution $\mathcal{R}_{\mathfrak{g}}$ in

$$\mathrm{Sh}^{\omega}(\mathfrak{g}) \widehat{\otimes} \mathrm{Sh}^{\omega}(\mathfrak{g}^*) \subset \mathrm{Sh}^{\omega}(D(\mathfrak{g})) \widehat{\otimes} \mathrm{Sh}^{\omega}(D(\mathfrak{g}))$$

of QYBE; $U_{\hbar}^{\omega}\mathfrak{g}$ and $U_{\hbar}^{\omega}\mathfrak{g}^*$ are the Hopf subalgebras of $\mathrm{Sh}^{\omega}(D(\mathfrak{g}))$ defined by $\mathcal{R}_{\mathfrak{g}}$. These are Hopf subalgebras of $U_{\hbar}^{\omega}(D(\mathfrak{g}))$, therefore $\mathcal{R}_{\mathfrak{g}}$ belongs to $(i_{\mathfrak{g},D(\mathfrak{g})})^U(U_{\hbar}^{\omega}\mathfrak{g}) \widehat{\otimes} (i_{\mathfrak{g}^*,D(\mathfrak{g})})^U(U_{\hbar}^{\omega}\mathfrak{g}^*)$.

If y belongs to $T_{\hbar}^{\omega}\mathfrak{g}$, and $x = \ell_{\mathfrak{g}}(y)$, then $x = \langle \mathcal{R}_{\mathfrak{g}}, \mathrm{id} \otimes y \rangle_{T_{\hbar}^{\omega}\mathfrak{g} \times \mathrm{Sh}^{\omega}(\mathfrak{g})}$, and $\Delta_{U_{\hbar}^{\omega}\mathfrak{g}}(x) = \langle \mathcal{R}_{\mathfrak{g}}^{(13)} \mathcal{R}_{\mathfrak{g}}^{(23)}, \mathrm{id} \otimes \mathrm{id} \otimes y \rangle_{T_{\hbar}^{\omega}\mathfrak{g} \times \mathrm{Sh}^{\omega}(\mathfrak{g})}$. Since $\mathcal{R}_{\mathfrak{g}}$ satisfies QYBE, and $U_{\hbar}^{\omega}\mathfrak{g}$ is the image of $\ell_{\mathfrak{g}}$, we have

$$\mathcal{R}_{\mathfrak{g}}((i_{\mathfrak{g},D(\mathfrak{g})})^U)^{\otimes 2}(\Delta_{U_{\hbar}^{\omega}(\mathfrak{g})}(x)) = ((i_{\mathfrak{g},D(\mathfrak{g})})^U)^{\otimes 2}(\Delta'_{U_{\hbar}^{\omega}(\mathfrak{g})}(x))\mathcal{R}_{\mathfrak{g}},$$

for any x in $U_{\hbar}^{\omega}\mathfrak{g}$. In the same way, one shows that $\mathcal{R}_{\mathfrak{g}}((i_{\mathfrak{g}^*,D(\mathfrak{g})})^U)^{\otimes 2}(\Delta_{U_{\hbar}^{\omega}(\mathfrak{g}^*)}(x)) = ((i_{\mathfrak{g}^*,D(\mathfrak{g})})^U)^{\otimes 2}(\Delta'_{U_{\hbar}^{\omega}(\mathfrak{g}^*)}(x))\mathcal{R}_{\mathfrak{g}}$, for any x in $U_{\hbar}^{\omega}\mathfrak{g}^*$. This proves that $\mathcal{R}_{\mathfrak{g}}\Delta_{U_{\hbar}^{\omega}(D(\mathfrak{g}))} = \Delta'_{U_{\hbar}^{\omega}(D(\mathfrak{g}))}\mathcal{R}_{\mathfrak{g}}$. So $(U_{\hbar}^{\omega}D(\mathfrak{g}), \mathcal{R}_{\mathfrak{g}})$ satisfies the axioms of the double QUE algebra of $U_{\hbar}^{\omega}\mathfrak{g}$ and may therefore be identified with the double of $U_{\hbar}^{\omega}\mathfrak{g}$.

The relation between $\mathcal{R}_{\mathfrak{g},can}$ and $\mathcal{R}_{\mathfrak{g}^*,can}$ follows from Remark 2 and the functoriality of $(\iota_{\mathfrak{g}} \otimes \iota_{\mathfrak{g}}^{-1})(\mathcal{R}_{\mathfrak{g},can})$ follows from that of $\mathcal{R}_{\mathfrak{g}}$. $\square$

Corollary 4.1. *For any $\omega \in \text{Ш}(\mathbb{K})$, the functor $\mathfrak{g} \mapsto U_{\hbar}^{\omega}\mathfrak{g}$ is a compatible quantization functor (see the Introduction).*

Proof. This follows from Propositions 4.3 and 4.4. $\square$

5. Proof of Theorem 0.2

5.1. **Identification of two quantizations of $F(\mathfrak{g}^*)$.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}^*$ is a Lie coalgebra. Let $F(\mathfrak{g}^*)$ be the free Lie algebra of $\mathfrak{g}^*$ (here $\mathfrak{g}^*$ is viewed as a vector space). Then the map $\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ dual to the bracket of $\mathfrak{g}$ extends to a unique cocycle map $\delta_{F(\mathfrak{g}^*)} : F(\mathfrak{g}^*) \to F(\mathfrak{g}^*)^{\otimes 2}$, which endows $F(\mathfrak{g}^*)$ with a Lie bialgebra structure. The corresponding Hopf-co-Poisson algebra is $(T(\mathfrak{g}^*), \delta_{T(\mathfrak{g}^*)})$. An element ω of $\mathcal{B}(\mathbb{K}[[\hbar]])$ being fixed, we now have two quantizations of this Hopf-co-Poisson algebra, namely $U_\hbar^\omega(F(\mathfrak{g}^*))$ and $T_\hbar^\omega(\mathfrak{g}^*)$.

Proposition 5.1. $U_\hbar^\omega(F(\mathfrak{g}^*))$ *and* $T_\hbar^\omega(\mathfrak{g}^*)$ *are canonically isomorphic.*

Proof. The inclusion of $\mathfrak{g}^*$ in $F(\mathfrak{g}^*)$ as its part of degree 1 is a morphism of Lie coalgebras. This morphism induces a morphism of Hopf algebras $i : T_\hbar^\omega(\mathfrak{g}^*) \to T_\hbar^\omega(F(\mathfrak{g}^*))$. Let us compose it with the projection $\ell_{F(\mathfrak{g}^*)} : T_\hbar^\omega(F(\mathfrak{g}^*)) \to U_\hbar^\omega(F(\mathfrak{g}^*))$. (Recall that $U_\hbar^\omega(F(\mathfrak{g}^*))$ is the image of $\ell_{F(\mathfrak{g}^*)}$, a Hopf subalgebra of $\mathrm{Sh}^\omega(F(\mathfrak{g}^*))$.) Then $\ell_{F(\mathfrak{g}^*)} \circ i$ is a morphism of Hopf algebras. Moreover, for any ξ in $\mathfrak{g}^*$, $i(\xi)$ is an element of $T_\hbar^\omega(F(\mathfrak{g}^*))$ in $\xi + o(\hbar)$, and the image of the latter element in $U_\hbar^\omega(F(\mathfrak{g}^*))$ is again in $\xi + o(\hbar)$. On the other hand, $U_\hbar^\omega(F(\mathfrak{g}^*))$ is a deformation of $T(\mathfrak{g}^*)$. The reduction mod $\hbar$ of the image of $\ell_{F(\mathfrak{g}^*)} \circ i$ contains $\mathfrak{g}^*$, so $\ell_{F(\mathfrak{g}^*)} \circ i$ is surjective. The reduction mod $\hbar$ of this map is the identity, so it is also a linear isomorphism. So we have shown that $\ell_{F(\mathfrak{g}^*)} \circ i$ is a Hopf algebra isomorphism from $T_\hbar^\omega(\mathfrak{g}^*)$ to $U_\hbar^\omega(\mathfrak{g}^*)$. This isomorphism is clearly functorial. $\square$

5.2. **Proof of Theorem 0.2.** Let us now prove Theorem 0.2. It follows from Corollary 4.1 that the map $\gamma_\mathbb{K}$ is a bijection from $\text{Ш}(\mathbb{K})$ to {universal quantization functors of the tensor algebras $T(\mathfrak{a})$}. In Theorem 4.1 and Proposition 4.1, we constructed a map $\alpha_\mathbb{K}$ from $\text{Ш}(\mathbb{K})$ to {universal quantization functors of the Lie bialgebras}.

Any universal quantization functor of Lie bialgebras may be restricted to the category of Lie bialgebras of the form $F(\mathfrak{a})$, $\mathfrak{a}$ a Lie coalgebra, and yields therefore a universal quantization functor of the tensor algebras $T(\mathfrak{a})$. Let us denote by $\beta'_\mathbb{K}$ the corresponding map from {universal quantization functors of the Lie bialgebras} to {universal quantization functors of the tensor algebras $T(\mathfrak{a})$}. Then the map $\beta_\mathbb{K}$ defined in the Introduction is $\gamma_\mathbb{K}^{-1} \circ \beta'_\mathbb{K}$. Proposition 5.1 implies that $\beta'_\mathbb{K} \circ \alpha_\mathbb{K} = \gamma_\mathbb{K}$. Therefore $\beta_\mathbb{K} \circ \alpha_\mathbb{K} = \mathrm{id}_{\text{Ш}(\mathbb{K})}$. This proves the first part of Theorem 0.2.

Let us prove the second part of this Theorem. It follows from Proposition 4.3 and Proposition 4.4 that the image of $\alpha_\mathbb{K}$ consists of quantization functors of Lie bialgebras that are compatible with the duals and doubles. Conversely, let us assume that Q is a quantization functor compatible with the duals and doubles, and let Q_0 be the restriction of Q to the Lie bialgebras of the form $F(\mathfrak{a})$ (so $Q_0 = \beta'_\mathbb{K}(Q)$). If $\mathfrak{a}$ is any Lie bialgebra, then the unique extension to $F(\mathfrak{a})$ of the identity map of the vector space $\mathfrak{a}$ to the Lie algebra $\mathfrak{a}$ induces a

Lie bialgebra morphism $F(\mathfrak{a}) \to \mathfrak{a}$. In the same way, we have a Lie bialgebra morphism $\mathfrak{a} \to F(\mathfrak{a}^*)^*$, and $\mathfrak{a}$ may be characterized as the image of the composed morphism $F(\mathfrak{a}) \to F(\mathfrak{a}^*)^*$. So $Q(\mathfrak{a})$ may be characterized as the image of the morphism $Q(F(\mathfrak{a})) \to Q(F(\mathfrak{a}^*)^*)$. Since Q is compatible with duals and restricts to Q_0, this is a Hopf algebra morphism from $Q_0(F(\mathfrak{a}))$ to $Q_0(F(\mathfrak{a}^*))^\vee$. Let us set $\omega = \gamma_{\mathbb{K}}^{-1}(Q_0)$, then this Hopf algebra morphism is the same as an element $R(\mathfrak{a})$ of $\mathrm{Sh}^\omega(\mathfrak{a}) \widehat{\otimes} \mathrm{Sh}^\omega(\mathfrak{a}^*)$, satisfying the rules (6). Therefore $R(\mathfrak{a})$ has the form $\mathcal{R}^\omega(\sigma_\mathfrak{a})$, with $\sigma_\mathfrak{a} \in \hbar(\mathfrak{a} \otimes \mathfrak{a}^*)[[\hbar]]$. Let us show that $R(\mathfrak{a})$ is the same as the image of the canonical R-matrix $R_{can}(\mathfrak{a})$ of $Q(D(\mathfrak{a}))$ by the tensor product of the injections $Q(\mathfrak{a}) \to Q_0(F(\mathfrak{a}^*))^\vee$ and $Q(\mathfrak{a}^*) \to Q_0(F(\mathfrak{a}))^\vee$.

By the properties of the quantum double, the identity map $Q(\mathfrak{a}) \to Q(\mathfrak{a})$ may be identified with the linear map $Q(\mathfrak{a}) \to Q(\mathfrak{a}^*)^\vee$ defined by $x \mapsto (\mathrm{id} \otimes x)(R_{can}(\mathfrak{a}))$. Since the canonical maps $Q(F\mathfrak{a}) \to Q(\mathfrak{a})$ and $Q(\mathfrak{a}^*)^\vee \to Q(F\mathfrak{a}^*)^\vee$ are respectively surjective and injective, the canonical map $Q(F\mathfrak{a}) \to Q(F\mathfrak{a}^*)^\vee$ may be defined by $x' \mapsto (\mathrm{id} \otimes x')(R'_{can}(\mathfrak{a}))$, where $R'_{can}(\mathfrak{a})$ is the image of $R_{can}(\mathfrak{a})$ by the tensor product of the injections $Q(\mathfrak{a}) \to Q_0(F(\mathfrak{a}^*))^\vee$ and $Q(\mathfrak{a}^*) \to Q_0(F(\mathfrak{a}))^\vee$. So $R'_{can}(\mathfrak{a}) = R(\mathfrak{a})$.

One can check that $R(\mathfrak{a})$ has the functoriality and duality properties $(\mathrm{Sh}^\omega(\phi) \otimes \mathrm{id})(R(\mathfrak{a})) = (\mathrm{id} \otimes \mathrm{Sh}^\omega(\phi^*))(R(\mathfrak{b}))$ and $R(\mathfrak{a}^*) = R(\mathfrak{a})^{(21)}$. It then follows from Remark 4.3 that the map $\mathfrak{a} \mapsto \sigma_\mathfrak{a}$ also has functoriality and duality properties.

Since we identified $R(\mathfrak{a})$ with the image of the R-matrix of the double $D(Q(\mathfrak{a}))$ of $Q(\mathfrak{a})$, and since $D(Q(\mathfrak{a}))$ identifies with $Q(D(\mathfrak{a}))$ and injects into $Q((D\mathfrak{a})^*)^\vee$, $R(\mathfrak{a})$ satisfies QYBE in the latter algebra. In Proposition 3.4, we defined a series $\rho^\omega = \sum_{n \geq 1} \rho_n^\omega$, such that if $r \in \hbar(\mathfrak{g} \otimes \mathfrak{g})[[\hbar]]$ is a solution of CYBE, then $\rho^\omega(r)$ is a solution of the Lie QYBE. The map $r \mapsto \rho^\omega(r)$ is bijective and one may show that it sets up a bijection between solutions of CYBE and of Lie QYBE. It follows that $(\rho^\omega)^{-1}(\sigma_\mathfrak{a})$ is a solution of CYBE. Let $\tau_\mathfrak{a}$ be the endomorphism of $\mathfrak{a}[[\hbar]]$, such that $(\rho^\omega)^{-1}(\sigma_\mathfrak{a}) = (\tau_\mathfrak{a} \otimes \mathrm{id})(r_\mathfrak{a})$. Since $(\rho^\omega)^{-1}(\sigma_\mathfrak{a})$ is expressed polynomially in terms of the structure constants of $\mathfrak{a}$, $\tau_\mathfrak{a}$ is a obtained by composition of tensor products of the bracket and cobracket map of $\mathfrak{a}$.

Lemma 5.1. *Let $\mathcal{H}_0$ be the set of functorial assignments $(\mathfrak{a} \mapsto \tau_\mathfrak{a})$, where $\mathfrak{a}$ runs over all finite-dimensional Lie bialgebras and for any $\mathfrak{a}$, $\tau_\mathfrak{a}$ belongs to $(\mathfrak{a} \otimes \mathfrak{a}^*)[[\hbar]]$, such that: 1) for any Lie algebra $\mathfrak{a}$, $\rho_\mathfrak{a}$ is a solution of CYBE (in $D(\mathfrak{a})^{\otimes 3}[[\hbar]]$), 2) $\rho_\mathfrak{a}$ satisfies $\tau_{\mathfrak{a}^*} = \tau_\mathfrak{a}^{(21)}$, 3) $\rho_\mathfrak{a}$ is equal to $r_\mathfrak{a}$ modulo $\hbar$ and is expressed polynomially in terms of the structure constants of $\mathfrak{a}$. Then the rule $(\mathfrak{a} \mapsto \rho_\mathfrak{a}) \mapsto (\mathfrak{a} \mapsto (\rho_\mathfrak{a} \otimes \mathrm{id})(r_\mathfrak{a}))$ defines a bijection from $\mathcal{G}_0$ to $\mathcal{H}_0$ ($\mathcal{G}_0$ has been defined in the Introduction).*

Proof. Let us set $r_\mathfrak{a} = \sum_{i \in I} a_i \otimes b_i$. Assume that $\tau_\mathfrak{a} \in (\mathfrak{a} \otimes \mathfrak{a}^*)[[\hbar]]$ is a solution of CYBE. Let $\rho_\mathfrak{a}$ be the endomorphism of $\mathfrak{a}[[\hbar]]$ such that $\tau_\mathfrak{a} = (\rho_\mathfrak{a} \otimes id)(r_\mathfrak{a})$. Then

we have also $\tau_{\mathfrak{a}} = (\mathrm{id} \otimes \rho_{\mathfrak{a}}^t)(r_{\mathfrak{a}})$, so $\rho_{\mathfrak{a}^*} = \rho_{\mathfrak{a}}^t$. We have

$$\llbracket \tau_{\mathfrak{a}}, \tau_{\mathfrak{a}} \rrbracket =$$

$$= \sum_{i,j \in I} [\rho_{\mathfrak{a}}(a_i), a_j] \otimes b_i \otimes \rho_{\mathfrak{a}}^t(b_j) + \rho(a_i) \otimes [b_i, a_j] \otimes \rho_{\mathfrak{a}}^t(b_j) + \rho(a_i) \otimes a_j \otimes [\rho_{\mathfrak{a}}^t(b_i), b_i]$$

$$= \sum_{i,j \in I} [\rho_{\mathfrak{a}}(a_i), a_j] \otimes b_i \otimes \rho_{\mathfrak{a}}^t(b_j) + \rho_{\mathfrak{a}}([a_j, a_i]) \otimes b_i \otimes \rho_{\mathfrak{a}}^t(b_j)$$

$$+ \rho_{\mathfrak{a}}(a_i) \otimes a_j \otimes \rho_{\mathfrak{a}}^t([b_j, b_i]) + \rho_{\mathfrak{a}}(a_i) \otimes a_j \otimes [\rho_{\mathfrak{a}}^t(b_i), b_i]$$

so $\tau_{\mathfrak{a}}$ satisfies the CYBE iff $\rho_{\mathfrak{a}}$ is such that $\rho_{\mathfrak{a}}([x, y]) = [\rho_{\mathfrak{a}}(x), y]$ for any pair x, y of elements of $\mathfrak{a}$ and $\delta_{\mathfrak{a}}(\rho_{\mathfrak{a}}(x)) = (\mathrm{id} \otimes \rho_{\mathfrak{a}})(\delta_{\mathfrak{a}}(x))$ for any element of $\mathfrak{a}$. The second condition is satisfied because $\rho_{\mathfrak{a}^*}([\xi, \eta]) = [\rho_{\mathfrak{a}^*}(\xi), \eta]$ for any pair ξ, η of elements of $\mathfrak{a}^*$. $\square$

End of proof of Theorem 0.2. It follows from this Lemma that $(\mathfrak{a} \mapsto \tau_{\mathfrak{a}})$ is an element of $\mathcal{G}_0$. Let us define $\varepsilon'_{\mathbb{K}}$ as the unique map from {compatible quantization functors of Lie bialgebras} to $\mathrm{III}(\mathbb{K}) \times \mathcal{G}_0$, such that $\varepsilon'_{\mathbb{K}}(Q) = (Q_0, \tau)$. It is clear how to reconstruct Q from (Q_0, τ), and that when τ is the neutral element of $\mathcal{G}_0$, this reconstruction coincides with the map $Q_0 \mapsto \alpha_{\mathbb{K}}(Q_0)$. $\varepsilon'_{\mathbb{K}}$ is therefore bijective, and we set $\varepsilon_{\mathbb{K}} = \varepsilon'^{-1}_{\mathbb{K}}$. $\square$

Remark 9. If $(\mathfrak{a} \mapsto \rho_{\mathfrak{a}})$ belong to $\mathcal{G}_0$, and $(\mathfrak{a}, [\ ,\ ], \delta_{\mathfrak{a}})$ is any Lie bialgebra, then $(\mathfrak{a}, [\ ,\ ], (\rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{a}}) \circ \delta_{\mathfrak{a}} \circ \rho_{\mathfrak{a}}^{-1})$ is again a Lie bialgebra. Indeed, set $r' = \sum_i \rho_{\mathfrak{a}}(a_i) \otimes b_i$. Since r' is a solution of CYBE in $D(\mathfrak{a})$, and r' belongs $(\mathfrak{a} \otimes \mathfrak{a}^*)[[\hbar]]$, if we set $\delta'(x) = [r', x \otimes 1 + 1 \otimes x]$, then $(\mathfrak{a}, [\ ,\ ]_{\mathfrak{a}}, \delta')$ is a Lie bialgebra. On the other hand, the rule $\rho_{\mathfrak{a}}([x, y]) = [\rho_{\mathfrak{a}}(x), y]$ implies that $\delta'(x) = (\rho_{\mathfrak{a}} \otimes \mathrm{id})(\delta_{\mathfrak{a}}(x))$ and the rules $\rho_{\mathfrak{a}^*}([\xi, \eta]) = [\rho_{\mathfrak{a}^*}(\xi), \eta]$ and $\rho_{\mathfrak{a}^*} = \rho_{\mathfrak{a}}^t$ imply that $(\rho_{\mathfrak{a}} \otimes \mathrm{id}) \circ \delta_{\mathfrak{a}} = \delta_{\mathfrak{a}} \circ \rho_{\mathfrak{a}}$. Since the image of $\delta_{\mathfrak{a}}$ is antisymmetric, we also have $(\mathrm{id} \otimes \rho_{\mathfrak{a}}) \circ \delta_{\mathfrak{a}} = \delta_{\mathfrak{a}} \circ \rho_{\mathfrak{a}}$, so $\delta' = (\rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{a}}) \circ \delta_{\mathfrak{a}} \circ \rho_{\mathfrak{a}}^{-1}$. So $\rho_{\mathfrak{a}}$ changes the Lie bialgebra structure of $\mathfrak{a}$ preserving both its Lie algebra and Lie coalgebra structures.

APPENDIX A. DEFORMATION OF SOLUTIONS OF CYBE (PROOF OF PROP. 3.5)

Let A be an associative algebra, and let $r_A \in A \otimes A$ be a solution of CYBE.

It is natural to look for a sequence $(\mathcal{R}_i)_{i \geq 0}$ of elements of $A \otimes A$, such that $\mathcal{R}_0 = 1, \mathcal{R}_1 = r_A$ and

$$\forall N \geq 0, \qquad \sum_{p,q,r \geq 0, p+q+r=N} \mathcal{R}_p^{12} \mathcal{R}_q^{13} \mathcal{R}_r^{23} = \sum_{p,q,r \geq 0, p+q+r=N} \mathcal{R}_r^{23} \mathcal{R}_q^{13} \mathcal{R}_p^{12}.$$

We call such a sequence $(\mathcal{R}_i)_{i \geq 0}$ a quantization of the CYBE solution r_A. If $(\mathcal{R}_i)_{i \geq 0}$ is such a quantization, then for any formal parameter t, $\sum_{i \geq 0} t^i \mathcal{R}_i$ is a solution of QYBE. Then if $(u_i)_{i \geq 0}$ is any sequence of elements of A such that $u_0 = 1$, the sequence $(\mathcal{R}'_i)_{i \geq 0}$ defined by $\sum_{i \geq 0} t^i \mathcal{R}'_i = u(\sum_{i \geq 0} t^i \mathcal{R}_i) u^{-1}$, where $u = \sum_{i \geq 0} t^i u_i$, is also a quantization of r_A. We will say that the sequences $(\mathcal{R}_i)_{i \geq 0}$ and $(\mathcal{R}'_i)_{i \geq 0}$ are equivalent.

One then seeks to solve inductively the above system of equations. More precisely, it can be written as follows

$$[\![r_A, \mathcal{R}_{N-1}]\!] = - \sum_{p,q,r \leq N-2, p+q+r=N} \mathcal{R}_p^{12} \mathcal{R}_q^{13} \mathcal{R}_r^{23} - \mathcal{R}_r^{23} \mathcal{R}_q^{13} \mathcal{R}_p^{12}, \qquad (34)$$

(we will call this equation the equation of order N) where we set

$$[\![r_A, R]\!] = [r_A^{12}, R^{13}] + [r_A^{12}, R^{23}] + [r_A^{23}, R^{23}] + [R^{12}, r_A^{13}] + [R^{12}, r_A^{23}] + [R^{23}, r_A^{23}].$$

$R \mapsto [\![r_A, R]\!]$ is therefore a linear map from $A^{\otimes 2}$ to $A^{\otimes 3}$ (the corresponding Lie algebraic map was called $\delta_{3,r}$ in Section 3.2.2).

On the other hand, let $\rho \mapsto \delta(r_A|\rho)$ be the linear map defined by

$$\delta(r_A|\rho) = [r_A^{12} + r_A^{13} + r_A^{14}, \rho^{234}] - [-r_A^{12} + r_A^{23} + r_A^{24}, \rho^{134}]$$
$$+ [-r_A^{13} - r_A^{23} + r_A^{34}, \rho^{124}] - [-r_A^{14} - r_A^{24} - r_A^{34}, \rho^{123}].$$

Then it follows from the fact that r_A is a solution of CYBE that $\delta(r_A|\cdot) \circ [\![r_A, \cdot]\!] = 0$ (this equality appears in the Lie coalgebra cohomology complex of A, endowed with the cobracket $\kappa(a) = [r_A, a \otimes 1 + 1 \otimes a]$). Therefore

$$\operatorname{Im}[\![r_A, \cdot]\!] \subset \operatorname{Ker} \delta(r_A|\cdot).$$

We will show

Theorem A.1. *Let us set*

$$H^1(A, r_A) = \operatorname{Ker}[\![r_A, \cdot]\!] / \operatorname{Im} \kappa, \qquad H^2(A, r_A) = \operatorname{Ker} \delta(r_A, |\cdot|) / \operatorname{Im}[\![r_A, \cdot]\!].$$

Then the equivalence classes of quantizations of r_A are obstructed by $H^2(A, r_A)$ and parametrized by $H^1(A, r_A)$.

In particular, for any pair (A, r_A) such that the homology $H^2(A, r_A)$ vanishes, the solution r_A can be quantized, the equivalence classes of its possible quantizations being parametrized by $\prod_{i \geq 2} H^1(A, r_A)$.

Proof of Theorem A.1. We will show that if $\mathcal{R}_1, \ldots, \mathcal{R}_{N-2}$ satisfy equations (34) at all orders $\leq N - 1$, then the right side of the equation (34) at order N is contained in $\operatorname{Ker} \delta(r_A|\cdot)$ (Theorem A.1).

We first prove

Lemma A.1. *For p integer ≥ 0, let $(R, \rho) \mapsto \delta_p(R, \rho)$ be the map from $A^{\otimes 2} \times A^{\otimes 3}$ to $A^{\otimes 4}$, defined by $\delta_0(R, \rho) = 0$, $\delta_p(R, \rho) = 0$ if $p \geq 4$ and*

$$\delta_1(R, \rho) = [R^{12} + R^{13} + R^{14}, \rho^{234}] - [-R^{12} + R^{23} + R^{24}, \rho^{134}]$$
$$+ [-R^{13} - R^{23} + R^{34}, \rho^{124}] - [-R^{14} - R^{24} - R^{34}, \rho^{123}],$$

$$\delta_2(R, \rho) = (R^{12}R^{13} + R^{12}R^{14} + R^{13}R^{14})\rho^{234} - \rho^{234}(R^{14}R^{13} + R^{14}R^{12} + R^{13}R^{12})$$
$$- R^{23}R^{24}\rho^{134} - (R^{23} + R^{24})\rho^{134}R^{12} + R^{12}\rho^{134}(R^{23} + R^{24}) + \rho^{134}R^{24}R^{23}$$
$$- R^{23}R^{13}\rho^{124} - (R^{13} + R^{23})\rho^{124}R^{34} + R^{34}\rho^{124}(R^{13} + R^{23}) + \rho^{124}R^{13}R^{23}$$
$$+ (R^{34}R^{24} + R^{34}R^{14} + R^{24}R^{14})\rho^{123} - \rho^{123}(R^{14}R^{24} + R^{14}R^{34} + R^{24}R^{34}),$$

and

$$\delta_3(R, \rho) = R^{12}R^{13}R^{14}\rho^{234} - \rho^{234}R^{14}R^{13}R^{12}$$
$$- R^{23}R^{24}\rho^{134}R^{12} + R^{12}\rho^{134}R^{24}R^{23}$$
$$- R^{23}R^{13}\rho^{124}R^{34} + R^{34}\rho^{124}R^{13}R^{23}$$
$$+ R^{34}R^{24}R^{14}\rho^{123} - \rho^{123}R^{14}R^{24}R^{34}.$$

Each δ_p is linear in ρ and homogeneous of degree p in R. Moreover, these maps satisfy the identity

$$\delta_p(R, R^{12}R^{13}R^{23} - R^{23}R^{13}R^{12}) + \delta_{p+1}(R, [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}]) = 0$$

for any integer $p \geq 0$.

Proof of Lemma A.1. This is a direct computation: for example, the identity for $p = 0$ follows from the Jacobi identity. □

As we said, the proof of Theorem A.1 reduces to the following statement.

Proposition A.1. *Let A be an algebra and r_A belong to $A \otimes A$. Let N be an integer ≥ 3 and assume that $\mathcal{R}_1, \ldots, \mathcal{R}_{N-2}$ in $A \otimes A$ satisfy $\mathcal{R}_1 = r_A$ and*

$$[\![r_A, \mathcal{R}_i]\!] = - \sum_{p+q+r=i+1, p,q,r>0} \mathcal{R}_p^{12} \mathcal{R}_q^{13} \mathcal{R}_r^{23} + \sum_{p+q+r=i+1, p,q,r>0} \mathcal{R}_r^{23} \mathcal{R}_q^{13} \mathcal{R}_p^{12}, \tag{35}$$

for $i = 1, \ldots, N - 2$ (in particular, r_A is a solution of CYBE). Then

$$\delta(r_A| \sum_{p+q+r=N, p,q,r>0} \mathcal{R}_p^{12} \mathcal{R}_q^{13} \mathcal{R}_r^{23} - \mathcal{R}_r^{23} \mathcal{R}_q^{13} \mathcal{R}_p^{12}) \tag{36}$$

is zero.

Proof of Proposition A.1. Let us define $\widetilde{\delta}_\alpha(R_1,\ldots,R_\alpha|\rho)$ as the coefficient of $t_1\cdots t_\alpha$ in $\frac{1}{\alpha!}\delta_\alpha(\sum_{\beta=1}^\alpha t_\beta R_\beta,\rho)$. Then $\widetilde{\delta}_\alpha(R_1,\ldots,R_\alpha|\rho)$ is the unique multilinear form in $(R_1,\ldots,R_\alpha)$, symmetric in these arguments, such that $\widetilde{\delta}_\alpha(R,\ldots,R|\rho) = \delta_\alpha(R|\rho)$.

Let us define $C(k,N,\alpha)$ as the subset of $\{1,\ldots,k\}^\alpha\times\{0,\ldots,k\}^3$ of $(\alpha+3)$uples $(k_1,\ldots,k_\alpha,p,q,r)$ such that $k_1+\ldots+k_\alpha+p+q+r=N$ and set

$$\mathcal{T}_k = \sum_{\alpha\geq 0}\ \sum_{(k_1,\ldots,k_\alpha,p,q,r)\in C(k,N,\alpha)}\widetilde{\delta}_\alpha(\mathcal{R}_{k_1},\ldots,\mathcal{R}_{k_\alpha}|\mathcal{R}_p^{12}\mathcal{R}_q^{13}\mathcal{R}_r^{23} - \mathcal{R}_r^{23}\mathcal{R}_q^{13}\mathcal{R}_p^{12}).$$

Then the fact that the $\mathcal{R}_i$ satisfy equations (35) implies that $\mathcal{T}_{N-3}$ is equal to

$$\delta(r_A|\sum_{p+q+r=N-1,p,q,r\leq N-3}\mathcal{R}_p^{12}\mathcal{R}_q^{13}\mathcal{R}_r^{23} - \mathcal{R}_r^{23}\mathcal{R}_q^{13}\mathcal{R}_p^{12}),$$

which is (36).

For $i=0,\ldots,3$, denote by $C_i(k,N,\alpha)$ the subset of $C(k,N,\alpha)$ of $(\alpha+3)$uples $(k_1,\ldots,k_\alpha,p,q,r)$ such that exactly i of the integers p,q,r is equal to zero. Then $C(k,N,\alpha)$ is the disjoint union of the $C_i(k,N,\alpha)$. Set

$$\mathcal{T}_{k,i} = \sum_{\alpha\geq 0}\ \sum_{(k_1,\ldots,k_\alpha,p,q,r)\in C_i(k,N,\alpha)}\widetilde{\delta}_\alpha(\mathcal{R}_{k_1},\ldots,\mathcal{R}_{k_\alpha}|\mathcal{R}_p^{12}\mathcal{R}_q^{13}\mathcal{R}_r^{23} - \mathcal{R}_r^{23}\mathcal{R}_q^{13}\mathcal{R}_p^{12}),$$

then we have $\mathcal{T}_k = \sum_{i=0}^3\mathcal{T}_{k,i}$. Since $\mathcal{T}_{k,2} = \mathcal{T}_{k,3} = 0$, we have

$$\mathcal{T}_k = \mathcal{T}_{k,0} + \mathcal{T}_{k,1}. \tag{37}$$

Now

$$\mathcal{T}_{k,0} = \sum_{\alpha\geq 0}\ \sum_{(k_1,\ldots,k_{\alpha+3})\in\{1,\ldots,k\}^{\alpha+3}|\sum_\beta k_\beta=N} \tag{38}$$
$$\widetilde{\delta}_\alpha(\mathcal{R}_{k_1},\ldots,\mathcal{R}_{k_\alpha}|\mathcal{R}_{k_{\alpha+1}}^{12}\mathcal{R}_{k_{\alpha+2}}^{13}\mathcal{R}_{k_{\alpha+3}}^{23} - \mathcal{R}_{k_{\alpha+3}}^{23}\mathcal{R}_{k_{\alpha+2}}^{13}\mathcal{R}_{k_{\alpha+1}}^{12}),$$

and

$$\mathcal{T}_{k,1} = \sum_{\alpha\geq 0}\ \sum_{(k_1,\ldots,k_{\alpha+3})\in\{1,\ldots,k\}^{\alpha+3}|\sum_\beta k_\beta=N} \tag{39}$$
$$\widetilde{\delta}_{\alpha+1}(\mathcal{R}_{k_1},\ldots,\mathcal{R}_{k_{\alpha+1}}|[\mathcal{R}_{k_{\alpha+2}}^{12},\mathcal{R}_{k_{\alpha+3}}^{13}] + [\mathcal{R}_{k_{\alpha+2}}^{12},\mathcal{R}_{k_{\alpha+3}}^{23}] + [\mathcal{R}_{k_{\alpha+2}}^{13},\mathcal{R}_{k_{\alpha+3}}^{23}]),$$

where we used the fact that $\delta_0 = 0$ to change α into $\alpha+1$ and where the first (resp., second, third) bracket corresponds to the subset of $C_1(N,k,\alpha)$ defined by the conditions $r=0, p\neq 0, q\neq 0$ (resp., $q=0, p\neq 0, r\neq 0$ and $p=0, q\neq 0, r\neq 0$).

Let us show that each $\mathcal{T}_k$ is equal to zero. Let us define, for $\underline{\nu} = (\nu_1,\cdots,\nu_k)$ a collection of integers ≥ 0 such that $\sum_{i=1}^k = \nu_i = \alpha+3$, $C(k,N,\underline{\nu})$ as the subset of $C(k,N,\alpha)$ of $(\alpha+3)$uples $(k_1,\ldots,k_{\alpha+3})$ of integers such that for any $i=1,\ldots,k$,

$\mathrm{card}\{\beta|k_\beta = i\} = \nu_i$. Set

$$\mathcal{T}_k^\nu = \sum_{(k_1,\dots,k_{\alpha+3})\in C(k,N,\alpha,\underline{\nu})} \Big(\widetilde{\delta}_\alpha(\mathcal{R}_{k_1},\dots,\mathcal{R}_{k_\alpha}|\mathcal{R}^{12}_{k_{\alpha+1}}\mathcal{R}^{13}_{k_{\alpha+2}}\mathcal{R}^{23}_{k_{\alpha+3}} - \mathcal{R}^{23}_{k_{\alpha+3}}\mathcal{R}^{13}_{k_{\alpha+2}}\mathcal{R}^{12}_{k_{\alpha+1}})$$

$$+ \widetilde{\delta}_{\alpha+1}(\mathcal{R}_{k_1},\dots,\mathcal{R}_{k_{\alpha+1}}|[\mathcal{R}^{12}_{k_{\alpha+2}},\mathcal{R}^{13}_{k_{\alpha+3}}] + [\mathcal{R}^{12}_{k_{\alpha+2}},\mathcal{R}^{23}_{k_{\alpha+3}}] + [\mathcal{R}^{13}_{k_{\alpha+2}},\mathcal{R}^{23}_{k_{\alpha+3}}])\Big).$$

Then it follows from (37) and the expressions (38) and (39) for $\mathcal{T}_{k,0}$ and $\mathcal{T}_{k,1}$ that $\mathcal{T}_k = \sum_{\underline{\nu}\in\mathbb{N}^k} \mathcal{T}_k^\nu$. Define the sequence $(k_1^\nu,\dots,k_{\alpha+3}^\nu)$ by the conditions that it is non-decreasing and that it belongs to $C(k,N,\underline{\nu})$. Then $\mathcal{T}_k^\nu$ is proportional to the symmetrization

$$\sum_{\sigma\in\mathfrak{S}_{\alpha+3}} \widetilde{\delta}_\alpha(\mathcal{R}_{k^\nu_{\sigma(1)}},\dots,\mathcal{R}_{k^\nu_{\sigma(\alpha)}}|\mathcal{R}^{12}_{k^\nu_{\sigma(\alpha+1)}}\mathcal{R}^{13}_{k^\nu_{\sigma(\alpha+2)}}\mathcal{R}^{23}_{k^\nu_{\sigma(\alpha+3)}} - \mathcal{R}^{23}_{k^\nu_{\sigma(\alpha+3)}}\mathcal{R}^{13}_{k^\nu_{\sigma(\alpha+2)}}\mathcal{R}^{12}_{k^\nu_{\sigma(\alpha+1)}})$$

$$+ \widetilde{\delta}_{\alpha+1}(\mathcal{R}_{k^\nu_{\sigma(1)}},\dots,\mathcal{R}_{k^\nu_{\sigma(\alpha+1)}}|[\mathcal{R}^{12}_{k^\nu_{\sigma(\alpha+2)}},\mathcal{R}^{13}_{k^\nu_{\sigma(\alpha+3)}}] + [\mathcal{R}^{12}_{k^\nu_{\sigma(\alpha+2)}},\mathcal{R}^{23}_{k^\nu_{\sigma(\alpha+3)}}] + [\mathcal{R}^{13}_{k^\nu_{\sigma(\alpha+2)}},\mathcal{R}^{23}_{k^\nu_{\sigma(\alpha+3)}}]).\tag{40}$$

We have now

Lemma A.2. *If α is an integer and $R_1,\dots,R_{\alpha+3}$ belong to $A\otimes A$, we have*

$$\sum_{\sigma\in\mathfrak{S}_{\alpha+3}} \widetilde{\delta}_\alpha(R_{\sigma(1)},\dots,R_{\sigma(\alpha)}|R^{12}_{\sigma(\alpha+1)}R^{13}_{\sigma(\alpha+2)}R^{23}_{\sigma(\alpha+3)} - R^{23}_{\sigma(\alpha+3)}R^{13}_{\sigma(\alpha+2)}R^{12}_{\sigma(\alpha+1)})$$

$$+ \widetilde{\delta}_{\alpha+1}(R_{\sigma(1)},\dots,R_{\sigma(\alpha+1)}|[R^{12}_{\sigma(\alpha+2)},R^{13}_{\sigma(\alpha+3)}] + [R^{12}_{\sigma(\alpha+2)},R^{23}_{\sigma(\alpha+3)}] + [R^{13}_{\sigma(\alpha+2)},R^{23}_{\sigma(\alpha+3)}]) = 0.$$

Proof of Lemma. The left side is multilinear and symmetric in variables $R_1,\dots,R_{\alpha+3}$; moreover, its value for $R_1 = \dots = R_{\alpha+3} = R$ is equal to

$$(\alpha+3)!\big(\delta_\alpha(R,R^{12}R^{13}R^{23}-R^{23}R^{13}R^{12})+\delta_{\alpha+1}(R,[R^{12},R^{13}]+[R^{12},R^{23}]+[R^{13},R^{23}])\big),$$

which is equal to zero. Therefore the left side is equal to zero. $\qquad\square$

End of proof of Proposition A.1. Since $\mathcal{T}_k^\nu$ is proportional to (40), the above Lemma implies that it is zero. It follows that for any k, $\mathcal{T}_k$ is equal to zero. In particular, $\mathcal{T}_{N-3} = 0$, which proves the Proposition. $\qquad\square$

Remark 10. When $N = 3$, equation (34) is written

$$[\![r_A, \mathcal{R}_2]\!] = -r_A^{12}r_A^{13}r_A^{23} + r_A^{23}r_A^{13}r_A^{12}.\tag{41}$$

One checks that the fact that r_A is a solution of CYBE implies that $\mathcal{R}_2 = \frac{1}{2}r_A^2$ is a solution of (41). Then if x is any element of A, then $\mathcal{R}_2 = \frac{1}{2}r_A^2 + \kappa(x)$ is also a solution of (41).

In Theorem 0.1, we construct a solution $\mathcal{R}(\rho(r_\mathfrak{a}))$ of QYBE. In that case, $A = \mathrm{Sh}(\mathfrak{a})[[\hbar]]$, r_A is the element $r_\mathfrak{a} \in \mathfrak{a}\otimes\mathfrak{a}$, viewed as an element of $A\otimes A$, and $\mathcal{R}_2 = \frac{1}{2}r_\mathfrak{a}^2 + \kappa(x)$, where $x = \frac{1}{2}\sum_{i\in I}(a_ib_i)$, and we write $r_\mathfrak{a} = \sum_{i\in I} a_i \otimes b_i$.

APPENDIX B. CONSTRUCTION OF $\mu_{Lie}^{p,q,r}$ (PROOF OF PROP. 3.2)

Let us denote by FA_n the part of the free algebra in n generators, homogeneous of degree one in each generator. In this Section, we also denote by FL_n the subspace of FA_n consisting of Lie elements (so $FL_n = Free_n$).

B.1. **Definition of the algebra** $(F^{(3)}, m_{F^{(3)}})$. When p, q, r are integers ≥ 0, we set

$$F_{pqr}^{(3)} = \left(FA_{q+r} \otimes (FA_p \otimes FA_q) \otimes FA_{p+r}\right)_{\mathfrak{S}_p \times \mathfrak{S}_q \times \mathfrak{S}_r} \quad \text{and} \quad F^{(3)} = \oplus_{p,q,r \geq 0} F_{pqr}^{(3)}.$$

Let us denote by $x_1^{(2)}, \ldots, x_q^{(2)}, x_1^{(3)}, \ldots, x_r^{(3)}$ the generators of FA_{q+r}, by $x_1^{(1)}, \ldots, x_p^{(1)}$ the generators of FA_p, by $y_1^{(2)}, \ldots, y_q^{(2)}$ the generators of FA_q, and by $y_1^{(1)}, \ldots, y_p^{(1)}$, $y_1^{(3)}, \ldots, y_r^{(3)}$ the generators of FA_{p+r}. Then the first and last set of generators are split in two subsets (e.g., the first subset of generators of FA_{q+r} is $x_1^{(2)}, \ldots, x_q^{(2)}$). The symmetric group $\mathfrak{S}_p$ (resp., $\mathfrak{S}_q$ and $\mathfrak{S}_r$) acts on $FA_{q+r} \otimes (FA_p \otimes FA_q) \otimes FA_{p+r}$ by simultaneously permuting the variables $(x_i^{(1)})_{i=1,\ldots,p}$ and $(y_i^{(1)})_{i=1,\ldots,p}$ (resp., the variables $(x_i^{(2)})_{i=1,\ldots,q}$ and $(y_i^{(2)})_{i=1,\ldots,q}$ and the variables $(x_i^{(3)})_{i=1,\ldots,r}$ and $(y_i^{(3)})_{i=1,\ldots,r}$).

If n, n' are integers ≥ 0, let us define $M_{n,n'}$ as the set of pairs of maps (c, c'), where c is a map from $\{1, \ldots, n'\}$ to $\{-\infty, 1, \ldots, n\}$ and c' is a map from $\{1, \ldots, n\}$ to $\{1, \ldots, n', \infty\}$, such that for any $k \in \{1, \ldots, n\}$, and any $k' \in \{1, \ldots, n'\}$, the inequalities

$$c(c'(k)) < k \quad \text{and} \quad c'(c(k')) > k' \tag{42}$$

hold whenever the left sides are defined. By convention, if k is any integer, $k < \infty$ and $-\infty < k$.

We are going to define a bilinear map $m_{F^{(3)}} : F^{(3)} \otimes F^{(3)} \to F^{(3)}$. If p, q and r are integers ≥ 0, there is a unique linear isomorphism $\xi \mapsto a_\xi$ from $FA_{q+r} \otimes FA_{p+r}$ to $F_{pqr}^{(3)}$, such that if $P \in FA_{q+r}, Q \in FA_{p+r}$ and $(\sigma, \tau) \in \mathfrak{S}_p \times \mathfrak{S}_q$, then

$$\begin{aligned} a_{P \otimes Q} =&P(x_1^{(2)}, \ldots, x_q^{(2)}, x_1^{(3)}, \ldots, x_r^{(3)}) \otimes x_1^{(1)} \ldots x_p^{(1)} y_1^{(2)} \ldots y_q^{(2)} \\ &\otimes Q(y_1^{(1)}, \ldots, y_p^{(1)}, y_1^{(3)}, \ldots, y_r^{(3)}). \end{aligned}$$

If p, q, r, and p', q', r' are integers ≥ 0, and if $P \in FA_{q+r}$, $Q \in FA_{p+r}$, $P' \in FA_{q'+r'}$, $Q' \in FA_{p'+r'}$, let us set

$$a_{(P,Q),(P',Q')} = \sum_{(c,c') \in M_{q,p'}} a_{(P,Q),(P',Q')}(c, c'),$$

where for any $(c, c') \in M_{q,p'}$, we set $\alpha = \mathrm{card}(c')^{-1}(\infty)$ and $\alpha' = \mathrm{card}\, c^{-1}(-\infty)$ and define $a_{(P,Q),(P',Q')}(c,c')$ as the element of $F^{(3)}_{p+\alpha',q'+\alpha,p'+q+r+r'-\alpha-\alpha'}$ given by

$$a_{(P,Q),(P',Q')}(c,c') =$$

$$\left(P\Big(\overset{\nearrow}{\prod_{j \in c^{-1}(1)}} \mathrm{ad}'(x^{(1)}_{p+j})(x^{(2)}_1), \dots, \overset{\nearrow}{\prod_{j \in c^{-1}(q)}} \mathrm{ad}'(x^{(1)}_{p+j})(x^{(2)}_q), x^{(3)}_1, \dots, x^{(3)}_r \right)$$

$$P'(x^{(2)}_{q+1}, \dots, x^{(2)}_{q+q'}, x^{(3)}_{r+1}, \dots, x^{(3)}_{r+r'})\Big)$$

$$\otimes \Big(\overset{\nearrow}{\prod_{i \in \{1,\dots,p\}}} x^{(1)}_i \overset{\nearrow}{\prod_{i' \in c^{-1}(\infty)}} x^{(1)}_{p+i'} \otimes \overset{\nearrow}{\prod_{j \in (c')^{-1}(\infty)}} y^{(2)}_j \overset{\nearrow}{\prod_{j' \in \{1,\dots,q'\}}} y^{(2)}_{q+j'} \Big)$$

$$\otimes \Big(Q(y^{(1)}_1, \dots, y^{(1)}_p, y^{(3)}_1, \dots, y^{(3)}_r)$$

$$Q'\Big(\overset{\searrow}{\prod_{j \in (c')^{-1}(1)}} \mathrm{ad}'(y^{(2)}_j)(y^{(1)}_{p+1}), \dots, \overset{\searrow}{\prod_{j \in (c')^{-1}(p')}} \mathrm{ad}'(y^{(2)}_j)(y^{(1)}_{p+p'}), y^{(3)}_{r+1}, \dots, y^{(3)}_{r+r'} \Big)\Big).$$

Let us explain the notation in this formula. The generators of the components of $F^{(3)}_{p+\alpha',q'+\alpha,p'+q+r+r'-\alpha-\alpha'}$ are denoted as follows:

– generators of $FA_{p'+q+q'+r+r'-\alpha'}$. First subset of generators: $x^{(2)}_i$, $i \in (c')^{-1}(\infty)$ and $x^{(2)}_{q+j'}$, $j' \in \{1, \dots, q'\}$. Second subset of generators: $x^{(1)}_{p+i'}$, $i' \in c^{-1}(\{1, \dots, q\})$, $x^{(2)}_j$, $j \in (c')^{-1}(\{1, \dots, q'\})$ and $x^{(3)}_k$, $k \in \{1, \dots, r+r'\}$

– the generators of $FA_{p+\alpha'}$ are $x^{(1)}_i$, $i \in \{1, \dots, p\}$ and $x^{(1)}_{p+i'}$, $i' \in c^{-1}(-\infty)$

– the generators of $FA_{p'+\alpha}$ are $y^{(2)}_j$, $j \in \{q+1, \dots, q+q'\}$ and $y^{(2)}_{j'}$, $j' \in (c')^{-1}(\infty)$

– generators of $FA_{p+p'+q+r+r'-\alpha}$. First subset of generators: $y^{(1)}_i$, $i \in \{1, \dots, p\}$, $y^{(2)}_{p+i'}$, $i' \in c^{-1}(-\infty)$. Second subset of generators: $y^{(1)}_{p+i'}$, $i' \in c^{-1}(\{1, \dots, q\})$, $y^{(2)}_j$, $j \in (c')^{-1}(\{1, \dots, p'\})$, $y^{(3)}_k$, $k \in \{1, \dots, r+r'\}$.

For x, y elements of an associative algebra, we set $\mathrm{ad}'(x)(y) = yx - xy$ (so $\mathrm{ad}' = -\mathrm{ad}$). If J is a finite ordered set of indices, and $J = \{j_1, \dots, j_r\}$, with $j_1 < \dots < j_r$, then $\prod^{\nearrow}_{j \in J} a_j$ and $\prod^{\searrow}_{j \in J} a_j$ denote the products of elements $a_{j_1} a_{j_2} \cdots a_{j_r}$ and $a_{j_r} a_{j_{r-1}} \cdots a_{j_1}$. So $\prod^{\nearrow}_{i \in \{1, \dots, n\}} \mathrm{ad}'(a_i)(a) = [[[a, a_n], a_{n-1}], \dots, a_1]$.

Let us set $\widetilde{F}_{pqr} = FA_{q+r} \otimes (FA_p \otimes FA_q) \otimes FA_{p+r}$ and $\widetilde{F} = \oplus_{p,q,r \geq 0} \widetilde{F}_{pqr}$. Then the rule $(P \otimes Q) \otimes (P' \otimes Q') \mapsto a_{(P,Q),(p',Q')}$ defines a linear map from $(FA_{q+r} \otimes Fa_{p+r}) \otimes (FA_{q'+r'} \otimes FA_{p'+r'})$ to $\widetilde{F}$, which is covariant with respect to the action of $\mathfrak{S}_r \times \mathfrak{S}_{r'}$. Therefore, its induces a linear map from $(FA_{q+r} \otimes Fa_{p+r})_{\mathfrak{S}_r} \otimes (FA_{q'+r'} \otimes FA_{p'+r'})_{\mathfrak{S}_{r'}}$ (which we identified with $F^{(3)}_{pqr} \otimes F^{(3)}_{p'q'r'}$) to $F^{(3)}$. This map may be extended by linearity in a unique way to a linear map $m_{F^{(3)}}$ from $F^{(3)} \otimes F^{(3)}$ to $F^{(3)}$.

B.2. **Associativity of** $m_{F^{(3)}}$**.** In this Section, we prove that $m_{F^{(3)}}$ is associative. For this, we first define composition operations on the sets of maps $M_{nn'}$ introduced above.

B.2.1. *The operations* $\mathrm{Comp}^{12,3}$ *and* $\mathrm{Comp}^{1,23}$. For any pair of integers (α, α'), define $M_{nn'}^{\alpha\alpha'}$ as the subset of $M_{n,n'}$ of all pairs (c, c') such that $\mathrm{card}\, c^{-1}(-\infty) = \alpha'$ and $\mathrm{card}(c')^{-1}(\infty) = \alpha$.

For any quadruple of integers (n, n', n'', n'''), let us also define $M_{n,n'|n'',n'''}$ as the subset of $M_{n+n',n''+n'''}$ of all pairs $(\widetilde{c}, \widetilde{c}')$ such that

$$\widetilde{c}(\{1, \dots, n'\}) \subset \{-\infty, 1, \dots, n\} \quad \text{and} \quad \widetilde{c}'(\{n+1, \dots, n+n''\}) \subset \{n'+1, \dots, n'+n''', \infty\}.$$

We define then two maps

$$\mathrm{Comp}^{12,3} : \coprod_{(\alpha,\alpha')\in\mathbb{N}^2} (M_{nn'}^{\alpha\alpha'} \times M_{\alpha+n'',n'''}) \to M_{n,n''|n',n'''}$$

and

$$\mathrm{Comp}^{1,23} : \coprod_{(\alpha'',\alpha''')\in\mathbb{N}^2} (M_{n,n'+\alpha'''} \times M_{n''n'''}^{\alpha''\alpha'''}) \to M_{n,n''|n',n'''}$$

in the following way. If $(c, c') \in M_{nn'}^{\alpha\alpha'}$ and $(c'', c''') \in M_{\alpha+n'',n'''}$, then

$$\mathrm{Comp}^{12,3}((c, c'), (c'', c''')) = (\widetilde{c}, \widetilde{c}'),$$

where $\widetilde{c}$ and $\widetilde{c}'$ are defined as follows. Let us denote the elements of $(c')^{-1}(\infty)$ by $i_1, \dots, i_\alpha$, where the sequence $(i_\beta)_{\beta=1,\dots,\alpha}$ is increasing. Then if $i' \in \{1, \dots, n'\}$, then $\widetilde{c}(i') = c(i')$; if $i''' \in \{1, \dots, n'''\}$ and $c''(i''') \in \{1, \dots, \alpha\}$, then $\widetilde{c}(n' + i''') = i_{c''(i''')}$; and if $c''(i''') \in \{\alpha+1, \dots, \alpha+n''\}$, then $\widetilde{c}(n' + i''') = (n - \alpha) + c''(i''')$. On the other hand, if $i \in \{1, \dots, n\} - (c')^{-1}(\infty)$, then $\widetilde{c}'(i) = c'(i)$, and for $\beta \in \{1, \dots, \alpha\}$, $\widetilde{c}'(i_\beta) = n' + c'''(\beta)$; and if $i'' \in \{1, \dots, n''\}$, then $\widetilde{c}'(n + i'') = n' + c'''(\alpha + i'')$.

By convention, if k is any integer, then $k + \infty = \infty$ and $-\infty + k = -\infty$.

In the same way, if $(d, d') \in M_{n,n'+\alpha''}$ and $(d'', d''') \in M_{n''n'''}^{\alpha''\alpha'''}$, then we set $\mathrm{Comp}^{1,23}((d, d'), (d'', d''')) = (\widetilde{d}, \widetilde{d}')$, where $\widetilde{d}$ and $\widetilde{d}'$ are defined as follows. Denote the elements of $(d'')^{-1}(-\infty)$ by $i_1, \dots, i_{\alpha''}$, where the sequence $(i_\beta)_{\beta=1,\dots,\alpha'''}$ is increasing. Then $\widetilde{d}(i') = d(i')$ if $i' \in \{1, \dots, n'\}$, $\widetilde{d}(n' + i_\beta) = d(n' + \beta)$ if $\beta \in \{1, \dots, \alpha'''\}$ and $\widetilde{d}(n' + i''') = n + d''(i''')$ if $i''' \in \{1, \dots, n'''\}$ and $i''' \notin (d''')^{-1}(\infty)$. On the other hand, if $i \in \{1, \dots, n\}$, then $\widetilde{d}'(i) = d'(i)$ if $d'(i) \in \{1, \dots, n'\}$, and $\widetilde{d}'(i) = n' + i_{d'(i)-n'}$ if $d'(i) \in \{n'+1, \dots, n'+\alpha'''\}$; and if $i'' \in \{1, \dots, n''\}$, then $\widetilde{d}'(n + i'') = n' + d'''(i'')$.

Lemma B.1. *The maps* $\mathrm{Comp}^{12,3}$ *and* $\mathrm{Comp}^{1,23}$ *are both bijective.*

Proof. One first checks that the pairs $(\widetilde{c}, \widetilde{c}')$ and $(\widetilde{d}, \widetilde{d}')$ defined above actually belong to $M_{n,n''|n',n'''}$. This is a direct verification. One also checks that each pair $(\widetilde{c}, \widetilde{c}')$ has a unique preimage by $\mathrm{Comp}^{12,3}$ and $\mathrm{Comp}^{1,23}$. For example, let

us describe $(\mathrm{Comp}^{12,3})^{-1}(\widetilde{c},\widetilde{c}') = ((c,c'),(c'',c'''))$. This is an element of $M_{nn'}^{\alpha\alpha'} \times M_{\alpha+n'',n'''}$, where

$$\alpha = \mathrm{card}\left(\widetilde{c}^{-1}(\{n'+1,\ldots,n'+n''',-\infty\}) \cap \{1,\ldots,n\}\right)$$

and $\alpha' = \mathrm{card}(\widetilde{c}'^{-1}(-\infty) \cap \{1,\ldots,n'\})$. Let us denote by $j_1,\ldots,j_\alpha$ the elements of $c^{-1}(\{n'+1,\ldots,n'+n''',-\infty\}) \cap \{1,\ldots,n\}$, where the sequence $(j_\beta)_{\beta=1,\ldots,\alpha}$ is increasing.

Then the pairs (c,c') and (c'',c''') are obtained as follows. If $j' \in \{1,\ldots,n'\}$, then $c(j') = \widetilde{c}(j')$. If $j''' \in \{1,\ldots,n'''\}$ and $\widetilde{c}(n'+j''') \in \{1,\ldots,n\}$, then since $\widetilde{c}'(\widetilde{c}(n'+j''')) > n'+j''' \geq n'+1$, $\widetilde{c}(n'+j''')$ belongs to $\{j_1,\ldots,j_\alpha\}$, and we define $c''(j''')$ as the index β such that $\widetilde{c}(n'+j''') = j_\beta$. If $j''' \in \{1,\ldots,n'''\}$ and $\widetilde{c}(n'+j''') \in \{n+1,\ldots,n+n'',-\infty\}$, then we set $c''(j''') = \widetilde{c}(n'+j''') + \alpha - n$.

If $j \in \{1,\ldots,n\}$, we define $c'(j)$ as $\widetilde{c}'(j)$ if $\widetilde{c}'(j) \in \{1,\ldots,n'\}$ and as ∞ else. For any $\beta \in \{1,\ldots,\alpha\}$, we define $c'''(\beta)$ as $\widetilde{c}'(j_\beta)-n'$, and if $j'' \in \{\alpha+1,\ldots,\alpha+n''\}$, we define $c'''(j'')$ as $\widetilde{c}'(j''+n-\alpha)-n'$. $\qquad\square$

B.2.2. *Associativity of $m_{F^{(3)}}$.*

Theorem B.1. *$m_{F^{(3)}}$ is associative.*

Proof. Let (p,q,r), (p',q',r') and (p'',q'',r'') be arbitrary triples of integers ≥ 0. Let P,P',P'' be elements of $FA_{q+r}, FA_{q'+r'}, FA_{q''+r''}$, let Q,Q',Q'' be elements of $FA_{p+r}, FA_{p'+r'}, FA_{p''+r''}$. It will be enough to prove that

$$m_F(a_{(P\otimes Q} \otimes a_{(P',Q'),(P'',Q'')}) = m_F(a_{(P,Q),(P',Q')} \otimes a_{P''\otimes Q''}). \tag{43}$$

The left side is a sum indexed by $\coprod_{(\alpha,\alpha')\in\mathbb{N}^2} M_{qp'}^{\alpha\alpha'} \times M_{\alpha+q',p''}$ and the right side is a sum indexed by $\coprod_{(\alpha'',\alpha''')\in\mathbb{N}^2} M_{q,p'+\alpha'''} \times M_{q'p''}^{\alpha''\alpha'''}$. Using maps $\mathrm{Comp}^{12,3}$ and $\mathrm{Comp}^{1,23}$, we transform both sums into the same expression

$$\sum_{(\widetilde{c},\widetilde{c}')\in M_{q,q'|p',p''}} a_{(P,Q),(P',Q'),(P'',Q'')}(\widetilde{c},\widetilde{c}'),$$

where we set $\beta = \mathrm{card}(\widetilde{c}')^{-1}(\infty)$, $\beta' = \mathrm{card}\,\widetilde{c}^{-1}(-\infty)$, and

$$a_{(P,Q),(P',Q'),(P'',Q'')}(\widetilde{c},\widetilde{c}')$$

is the element of $F^{(3)}_{p+\beta,q''+\beta',p'+p''+q+q'+r+r'+r''-\beta-\beta'}$ given by

$$a_{(P,Q),(P',Q'),(P'',Q'')}(\widetilde{c},\widetilde{c}') = \tag{44}$$

$$\big(P\big(\prod_{i\in\widetilde{c}^{-1}(1)}^{\nearrow}\mathrm{ad}'(x^{(1)}_{p+i})(x^{(2)}_1),\dots,\prod_{i\in\widetilde{c}^{-1}(q)}^{\nearrow}\mathrm{ad}'(x^{(1)}_{p+i})(x^{(2)}_q),x^{(3)}_1,\dots,x^{(3)}_r)$$

$$P'\big(\prod_{i\in\widetilde{c}^{-1}(q+1)}^{\nearrow}\mathrm{ad}'(x^{(1)}_{p+i})(x^{(2)}_{q+1}),\dots,\prod_{i\in\widetilde{c}^{-1}(q+q')}^{\nearrow}\mathrm{ad}'(x^{(1)}_{p+i})(x^{(2)}_{q+q'}),x^{(3)}_{r+1},\dots,x^{(3)}_{r+r'})$$

$$P''(x^{(2)}_{q+q'+1},\dots,x^{(2)}_{q+q'+q''},x^{(3)}_{r+r'+1},\dots,x^{(3)}_{r+r'+r''})\big)$$

$$\otimes\big(\prod_{i\in\{1,\dots,p\}}^{\nearrow}x^{(1)}_i\prod_{i'\in\widetilde{c}^{-1}(-\infty)}^{\nearrow}x^{(1)}_{p+i'}\otimes\prod_{j\in(\widetilde{c}')^{-1}(\infty)}^{\nearrow}y^{(2)}_j\prod_{j''\in\{1,\dots,q''\}}^{\nearrow}y^{(2)}_{q+q'+j''}\big)$$

$$\otimes\big(Q(y^{(1)}_1,\dots,y^{(1)}_p,y^{(3)}_1,\dots,y^{(3)}_r)$$

$$Q'\big(\prod_{j\in(\widetilde{c}')^{-1}(1)}^{\searrow}\mathrm{ad}'(y^{(2)}_j)(y^{(1)}_{p+1}),\dots,\prod_{j\in(\widetilde{c}')^{-1}(p')}^{\searrow}\mathrm{ad}'(y^{(2)}_j)(y^{(1)}_{p+p'}),y^{(3)}_{r+1},\dots,y^{(3)}_{r+r'})$$

$$Q''\big(\prod_{j\in(\widetilde{c}')^{-1}(p'+1)}^{\searrow}\mathrm{ad}'(y^{(2)}_j)(y^{(1)}_{p+p'+1}),\dots,\prod_{j\in(\widetilde{c}')^{-1}(p'+p'')}^{\searrow}$$

$$\mathrm{ad}'(y^{(2)}_j)(y^{(1)}_{p+p'+p''}),y^{(3)}_{r+r'+1},\dots,y^{(3)}_{r+r'+r''})\big).$$

In this formula, the generators of the factors of $F^{(3)}_{p+\beta,q''+\beta',p'+p''+q+q'+r+r'+r''-\beta-\beta'}$ are denoted as follows:

– generators of $F^{(3)}_{p'+p''+q+q'+q''+r+r'+r''-\beta}$. First subset of generators: $x^{(2)}_j$, $j \in (\widetilde{c}')^{-1}(\infty)$, $x^{(2)}_{q+q'+j''}$, $j'' \in \{1,\dots,q''\}$. Second subset of generators: $x^{(1)}_{p+j'}$, $j' \in \widetilde{c}^{-1}(\{1,\dots,q+q'\})$, $x^{(2)}_j$, $j \in (\widetilde{c})^{-1}(\{1,\dots,p'+p''\})$, $x^{(3)}_k$, $k \in \{1,\dots,r+r'+r''\}$.

– generators of $F^{(3)}_{p+\beta}$: $x^{(1)}_i$, $i \in \{1,\dots,p\}$ and $x^{(1)}_{p+i'}$, $i' \in \widetilde{c}^{-1}(\infty)$

– generators of $F^{(3)}_{q''+\beta'}$: $y^{(2)}_j$, $j \in (\widetilde{c}')^{-1}(\infty)$, and $y^{(2)}_{q+q'+j''}$, $j'' \in \{1,\dots,q''\}$

– generators of $F^{(3)}_{p+p'+p''+q+q'+r+r'+r''-\beta'}$. First subset of generators: $y^{(1)}_i$, $i \in \{1,\dots,p\}$, and $y^{(1)}_{p+i'}$, $i' \in \widetilde{c}^{-1}(\infty)$. Second subset of generators: $y^{(1)}_{p+i'}$, $i' \in \widetilde{c}^{-1}(\{1,\dots,q+q'\})$, $y^{(2)}_i$, $i \in (\widetilde{c})^{-1}(\{1,\dots,p'+p''\})$, and $y^{(3)}_i$, $i \in \{1,\dots,r+r'+r''\}$.

Since each side of (43) is equal to (44), equation (43) is satisfied. $\square$

B.3. **Universal properties of** $(F^{(3)}, m_{F^{(3)}})$. If p, q, r are integers ≥ 0, a basis of $F_{pqr}^{(3)}$ consists of the

$$\prod_{k=1}^{q+r} z_{\sigma(k)} \otimes (\prod_{i=1}^{p} x_i^{(1)} \otimes \prod_{j=1}^{q} y_j^{(2)}) \otimes \prod_{l=1}^{p+r} t_{\tau(l)},$$

where $\sigma \in \mathfrak{S}_{q+r}, \tau \in \mathfrak{S}_{p+r}, (z_1, \dots, z_{q+r}) = (x_1^{(2)}, \dots, x_q^{(2)}, x_1^{(3)}, \dots, x_r^{(3)}), (t_1, \dots, t_{p+r}) = (y_1^{(1)}, \dots, y_p^{(1)}, y_1^{(3)}, \dots, y_r^{(3)})$, and σ preserves the order of the r last elements of $\{1, \dots, q+r\}$.

If $\mathfrak{a}$ is a Lie algebra and $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution of CYBE, then there are unique maps

$$\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\text{tensor}} : F^{(3)} \to T\mathfrak{a} \otimes U\mathfrak{a} \otimes T\mathfrak{a} \quad \text{and} \quad \kappa_{\mathfrak{a}, r_{\mathfrak{a}}} : F^{(3)} \to U\mathfrak{a}^{\otimes 3},$$

where $T\mathfrak{a}$ is the tensor algebra of $\mathfrak{a}$, such that if $r_{\mathfrak{a}} = \sum_{i \in I} a_i \otimes b_i$, then $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\text{tensor}}$ maps $F_{pqr}^{(3)}$ to $\mathfrak{a}^{\otimes q+r} \otimes U\mathfrak{a} \otimes \mathfrak{a}^{\otimes p+r}$ in such a way that

$$\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\text{tensor}} (\prod_{k=1}^{q+r} z_{\sigma(k)} \otimes (\prod_{i=1}^{p} x_i^{(1)} \otimes \prod_{j=1}^{q} y_j^{(2)}) \otimes \prod_{l=1}^{p+r} t_{\tau(l)})$$

$$= \sum_{\alpha_1, \dots, \alpha_{p+q+r} \in I} (\bigotimes_{k=1}^{q+r} a(\alpha_{p+\sigma(k)})) \otimes \prod_{i=1}^{p} a(\alpha_i) \prod_{j=1}^{q} b(\alpha_{p+j}) \otimes (\bigotimes_{j=1}^{p+r} b(\beta(\tau, l))),$$

where $\beta(\tau, l)$ is equal to $\tau(l)$ if $\tau(l) \leq p$ and to $q + \tau(l)$ else, and $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}$ is the composition of $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\text{tensor}}$ with the projection map $T\mathfrak{a} \to U\mathfrak{a}$.

Proposition B.1. *For any pair* $(\mathfrak{a}, r_{\mathfrak{a}})$ *of a Lie algebra and a solution of CYBE,* $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\text{tensor}}$ *is an algebra morphism from* $(F^{(3)}, m_{F^{(3)}})$ *to* $T\mathfrak{a} \otimes U\mathfrak{a} \otimes T\mathfrak{a}$, *and* $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}$ *is an algebra morphism from* $(F^{(3)}, m_{F^{(3)}})$ *to* $U\mathfrak{a}^{\otimes 3}$.

Proof. The proof of the statement on $\kappa_{\mathfrak{a}, r_A}^{\text{tensor}}$ is by a double induction on (q, q'). The statement on $\kappa_{\mathfrak{a}, r_A}$ follows immediately. $\qquad\square$

B.4. **The normal ordering map.** For any triple of integers $p, q, r \geq 0$, let us form the tensor product $FA_{q+r} \otimes FA_{p+q} \otimes FA_{p+r}$; we denote the generators of its factors as follows
   – generators of FA_{q+r}: $x_1^{(2)}, \dots, x_q^{(2)}, x_1^{(3)}, \dots, x_r^{(3)}$
   – generators of FA_{p+q}: $y_1^{(2)}, \dots, y_q^{(2)}, x_1^{(1)}, \dots, x_p^{(1)}$
   – generators of FA_{p+r}: $y_1^{(1)}, \dots, y_p^{(1)}, y_1^{(3)}, \dots, y_r^{(3)}$.
   Each set of generators is split in two subsets (e.g., the first subset of generators of FA_{q+r} is $x_1^{(2)}, \dots, x_q^{(2)}$ and the second subset is $x_1^{(3)}, \dots, x_r^{(3)}$).
   Then the group $\mathfrak{S}_p \times \mathfrak{S}_q \times \mathfrak{S}_r$ acts on this tensor product by simultaneous permutations of the three pairs of subsets of generators. We will set

$$G_{pqr}^{(3)} = (FA_{q+r} \otimes FA_{p+q} \otimes FA_{p+r})_{\mathfrak{S}_p \times \mathfrak{S}_q \times \mathfrak{S}_r}.$$

Let $\mathrm{Part}_{p,q}$ be the set of pairs of partitions $(\underline{p}, \underline{q})$, where $\underline{p} = (p_1, \ldots, p_\lambda)$ and $\underline{q} = (q_1, \ldots, q_\lambda)$ are partitions of p and q, such that $p_\lambda \geq 0$, $q_\lambda \geq 0$ and $p_\nu > 0$, $q_{\nu'} > 0$ when $\nu \neq \lambda$ and $\nu' \neq 1$. For any $(\underline{p}, \underline{q}) \in \mathrm{Part}_{p,q}$, let us denote by $G^{\underline{p},\underline{q}}_{pqr}$ the image of

$$FA_{q+r} \otimes \prod_{\nu=1}^{\lambda} \left(\prod_{j=1}^{q_\nu} y^{(2)}_{q_1+\cdots+q_{\nu-1}+j} \prod_{i=1}^{p_\nu} x^{(1)}_{p_1+\cdots+p_{\nu-1}+i} \right) \otimes FA_{p+r}$$

in $G^{(3)}_{pqr}$. Then we have

$$G^{(3)}_{pqr} = \bigoplus_{(\underline{p},\underline{q}) \in \mathrm{Part}_{p,q}} G^{\underline{p},\underline{q}}_{pqr}.$$

Let us define $M_{\underline{q},\underline{p}}$ as the subset of M_{qp} consisting of all pairs of maps (c, c'), where $c : \{1, \ldots, q\} \to \{-\infty, 1, \ldots, p\}$ and $c' : \{1, \ldots, p\} \to \{1, \ldots, q, \infty\}$ satisfy (42) and are such that for any $\nu = 1, \ldots, \lambda$,

$$c(\{1, \ldots, \sum_{i=1}^{\nu} q_i\}) \subset \{-\infty, 1, \ldots, \sum_{i=1}^{\nu} p_i\}$$

and

$$c'(\{\sum_{i=1}^{\lambda} p_i + 1, \ldots, p\}) \subset \{\sum_{i=\nu}^{\lambda} q_i + 1, \ldots, q, \infty\}.$$

Let us define the map

$$\mu^{\underline{p},\underline{q}}_{pqr} : G^{\underline{p},\underline{q}}_{pqr} \to F^{(3)}$$

as follows. If $P \in FA_{q+r}$, $Q \in FA_{p+r}$, let us set

$$\mu^{\underline{p},\underline{q}}_{pqr}\left(P \otimes \prod_{\nu=1}^{\lambda} \left(\prod_{j=1}^{q_\nu} y^{(2)}_{q_1+\cdots+q_{\nu-1}+j} \prod_{i=1}^{p_\nu} x^{(1)}_{p_1+\cdots+p_{\nu-1}+i} \right) \otimes Q\right) = \sum_{(c,c') \in M_{\underline{q},\underline{p}}} \alpha^{\underline{p},\underline{q}}_{p,q}(P,Q)(c,c');$$

we set $\beta = \mathrm{card}(c')^{-1}(\infty)$ and $\beta' = \mathrm{card}\, c^{-1}(-\infty)$, and define $\alpha^{\underline{p},\underline{q}}_{p,q}(P, Q, \gamma, \delta)(c, c')$ as the element of $F^{(3)}_{p-\beta, q-\beta', r+\beta+\beta'}$ equal to

$$\alpha^{\underline{p},\underline{q}}_{p,q}(P,Q)(c,c')$$

$$= P\left(\overset{\nearrow}{\prod_{i_1 \in (c')^{-1}(1)}} \mathrm{ad}'(x^{(1)}_{i_1})(x^{(2)}_1), \ldots, \overset{\nearrow}{\prod_{i_q \in (c')^{-1}(q)}} \mathrm{ad}'(x^{(1)}_{i_q})(x^{(2)}_q), x^{(3)}_1, \ldots, x^{(3)}_r \right)$$

$$\otimes \left(\overset{\nearrow}{\prod_{i \in (c')^{-1}(\infty)}} x^{(1)}_i \otimes \overset{\nearrow}{\prod_{j \in c^{-1}(-\infty)}} y^{(2)}_j \right)$$

$$\otimes Q\left(\overset{\searrow}{\prod_{j_1 \in c^{-1}(1)}} \mathrm{ad}'(y^{(2)}_{j_1})(y^{(1)}_1), \ldots, \overset{\searrow}{\prod_{j_p \in c^{-1}(p)}} \mathrm{ad}'(y^{(2)}_{j_p})(y^{(1)}_p), y^{(3)}_1, \ldots, y^{(3)}_r \right)$$

where the generators of the components of $F^{(3)}_{p-\beta,q-\beta',r+\beta+\beta'}$ are denoted as follows

– generators of $FA_{q+r+\beta}$. First set of generators: $x^{(2)}_j$, $j \in c^{-1}(-\infty)$. Second set of generators: $x^{(1)}_j$, $j \in (c')^{-1}(\{1,\ldots,q\})$, $x^{(2)}_i$, $i \in c^{-1}(\{1,\ldots,p\})$, and $x^{(3)}_1,\ldots,x^{(3)}_r$

– generators of $FA_{p-\beta}$: $x^{(1)}_i$, $i \in (c')^{-1}(\infty)$.

– generators of $FA_{q-\beta'}$: $y^{(2)}_j$, $j \in c^{-1}(-\infty)$.

– generators of $FA_{p+r+\beta'}$. First set of generators: $y^{(1)}_i$, $i \in (c')^{-1}(\infty)$. Second set of generators: $y^{(1)}_j$, $j \in (c')^{-1}(\{1,\ldots,q\})$, $y^{(2)}_i$, $i \in c^{-1}(\{1,\ldots,p\})$, and $y^{(3)}_1,\ldots,y^{(3)}_r$.

Let $\mathfrak{a}$ be a Lie algebra and let $r_\mathfrak{a} \in \mathfrak{a} \otimes \mathfrak{a}$ be a solution of CYBE. Elements of the form

$$z_{\sigma(1)}\cdots z_{\sigma(q+r)}\prod_{\nu=1}^{\lambda}\Big(\prod_{j=1}^{q_\nu} y^{(2)}_{q_1+\cdots+q_{\nu-1}+j}\prod_{i=1}^{p_\nu} x^{(1)}_{p_1+\cdots+p_{\nu-1}+i}\Big)\otimes w_{\tau(1)}\cdots w_{\tau(p+r)},$$

form a generating family of $G^{(3)}_{pqr}$, where we set $(z_1,\ldots,z_{q+r}) = (x^{(2)}_1,\ldots,x^{(2)}_q,x^{(3)}_1,\ldots,x^{(3)}_r)$, $(w_1,\ldots,w_{p+r}) = (y^{(1)}_1,\ldots,y^{(1)}_p,y^{(3)}_1,\ldots,y^{(3)}_r)$, and $(\underline{p},\underline{q})$ belongs to $\mathrm{Part}_{p,q}$ and (σ,τ) belongs to $\mathfrak{S}_{q+r}\times\mathfrak{S}_{p+r}$. Then there is a unique map

$$\gamma^{(\mathfrak{a},r_\mathfrak{a})}_{pqr} : G^{(3)}_{pqr} \to T\mathfrak{a}\otimes U\mathfrak{a}\otimes T\mathfrak{a},$$

such that for any $(\underline{p},\underline{q}) \in \mathrm{Part}_{p,q}$ and any $\sigma \in \mathfrak{S}_{q+r}$, $\tau \in \mathfrak{S}_{p+r}$,

$$\gamma^{(\mathfrak{a},r_\mathfrak{a})}_{pqr}\big(z_{\sigma(1)}\cdots z_{\sigma(q+r)}\otimes\prod_{\nu=1}^{\lambda}\big(\prod_{j=1}^{q_\nu} y^{(2)}_{q_1+\cdots+q_{\nu-1}+j}\prod_{i=1}^{p_\nu} x^{(1)}_{p_1+\cdots+p_{\nu-1}+i}\big)\otimes w_{\tau(1)}\cdots w_{\tau(p+r)}\big)$$

$$= \sum_{\alpha(1),\ldots,\alpha(p+q+r)\in I} a(\alpha(p+\sigma(1)))\cdots a(\alpha(p+\sigma(q+r)))$$

$$\otimes\prod_{\nu=1}^{\lambda}\big(\prod_{j=1}^{q_\nu} b(\alpha(q_1+\cdots+q_{\nu-1}+j))\prod_{i=1}^{p_\nu} a(\alpha(p_1+\cdots+p_{\nu-1}+i))\big)\otimes b(\epsilon(1))\cdots b(\epsilon(p+r)),$$

where $\epsilon(i) = \tau(j)$ if $\tau(j) \le p$ and $\epsilon(j) = q + \tau(j)$ if $\tau(j) > p$, and we set $r_\mathfrak{a} = \sum_{\alpha\in I} a(\alpha)\otimes b(\alpha)$.

Let us define $\mu_{pqr} : G^{(3)}_{pqr} \to F^{(3)}$ as the direct sum of the maps $\mu^{p,q}_{pqr}$. Then μ_{pqr} has the following property.

Proposition B.2. *We have* $\kappa_{\mathfrak{a},r_\mathfrak{a}} \circ \mu_{pqr} = \gamma^{(\mathfrak{a},r_\mathfrak{a})}_{pqr}$.

Proof. By induction. $\square$

Remark 11. The direct sum $\mu^{(3)} = \oplus_{(p,q,r)\in\mathbb{N}^3}\mu_{pqr}$ may therefore be viewed as a universal version of the normal ordering of expressions in a solution $r_\mathfrak{a}$ of CYBE. We will sometimes identify elements of G_{pqr} with their images in $F^{(3)}$ by $\mu^{(3)}$.

B.5. The CYBE identity in $F^{(3)}$.

We have the following identity in $F^{(3)}$

$$\mu^{(3)}(x_1^{(1)}\otimes[y_1^{(1)},x_1^{(2)}]\otimes x_1^{(2)}) = -\mu^{(3)}([x_1^{(1)},x_1^{(3)}]\otimes y_1^{(1)}\otimes y_1^{(3)})-\mu^{(3)}(x_1^{(3)}\otimes x_1^{(2)}\otimes[x_1^{(2)},y_1^{(3)}]),$$

in which the first expression belongs to $\mu^{(3)}(G_{110}^{(3)})$, the second to $\mu^{(3)}(G_{101}^{(3)})$ and the third to $\mu^{(3)}(G_{011}^{(3)})$. (The image of this identity by any map $\kappa_{\mathfrak{a},r_{\mathfrak{a}}}$ simply expresses the fact that $r_{\mathfrak{a}}$ satisfies CYBE.)

Let us define a map $\mathrm{conc}_{G^{(3)}} : G^{(3)} \otimes G^{(3)} \to G^{(3)}$ as follows. $\mathrm{conc}_{G^{(3)}}$ maps $G_{pqr}^{(3)} \otimes G_{p'q'r'}^{(3)}$ to $G_{p+p',q+q',r+r'}^{(3)}$; if P,Q,R,P',Q',R' belong to FA_{q+r}, FA_{p+q}, FA_{p+r}, $FA_{q'+r'}$, $FA_{p'+q'}$, $FA_{p'+r'}$, then

$$\mathrm{conc}_{G^{(3)}}((P \otimes Q \otimes R) \otimes (P' \otimes Q' \otimes R'))$$
$$= P(x_1^{(2)},\dots,x_q^{(2)},x_1^{(3)},\dots,x_r^{(3)})P'(x_{q+1}^{(2)},\dots,x_{q+q'}^{(2)},x_{r+1}^{(3)},\dots,x_{r+r'}^{(3)})$$
$$\otimes Q(x_1^{(1)},\dots,x_p^{(1)},y_1^{(2)},\dots,y_q^{(2)})Q'(x_{p+1}^{(1)},\dots,x_{p+p'}^{(1)},y_{q+1}^{(2)},\dots,y_{q+q'}^{(2)})$$
$$\otimes R(y_1^{(1)},\dots,y_p^{(1)},y_1^{(3)},\dots,y_r^{(3)})R'(y_{p+1}^{(1)},\dots,y_{p+p'}^{(1)},y_{r+1}^{(3)},\dots,y_{r+r'}^{(3)}).$$

Then we have

Lemma B.2. $(G^{(3)},\mathrm{conc}_{G^{(3)}})$ *is an algebra and μ is an algebra morphism from* $(G^{(3)},\mathrm{conc}_{G^{(3)}})$ *to* $(F^{(3)},m_{F^{(3)}})$.

This follows from analysis of the behavior the spaces of maps M_{pqr}^{pq}.

We have then

Lemma B.3. *If $P \in FA_{q+r}$, $Q \in FA_{p+q-1}$ and $R \in FA_{p+r}$, we have the identity*

$$\mu^{(3)}\big(P(x_1^{(2)},\dots,x_q^{(2)},x_1^{(3)},\dots,x_r^{(3)}) \otimes Q(y_1^{(2)},\dots,[y_j^{(2)},x_i^{(1)}],\dots,y_q^{(2)},x_1^{(1)},\dots,\check{i},\dots,x_p^{(1)})$$
$$\otimes R(y_1^{(1)},\dots,y_p^{(1)},y_1^{(3)},\dots,y_q^{(3)}))\big)$$
$$= -\mu^{(3)}\big(P(x_1^{(2)},\dots,[x_i^{(1)},x_j^{(2)}]\dots,x_q^{(2)},x_1^{(3)},\dots,x_r^{(3)}) \otimes Q(y_1^{(2)},\dots,y_q^{(2)},x_1^{(1)},\dots,\check{i},\dots,x_p^{(1)})$$
$$\otimes R(y_1^{(1)},\dots,y_p^{(1)},y_1^{(3)},\dots,y_q^{(3)}))\big)$$
$$- \mu^{(3)}\big(P(x_1^{(2)},\dots,x_q^{(2)},x_1^{(3)},\dots,x_r^{(3)}) \otimes Q(y_1^{(2)},\dots,x_i^{(1)},\dots,y_q^{(2)},x_1^{(1)},\dots,\check{i},\dots,x_p^{(1)})$$
$$\otimes R(y_1^{(1)},\dots,[y_i^{(1)},y_j^{(2)}],\dots,y_p^{(1)},y_1^{(3)},\dots,y_q^{(3)}))\big),$$

where the arguments of μ belong to $G_{pqr}^{(3)}$, $G_{p-1,q,r+1}^{(3)}$ and $G_{p,q-1,r+1}^{(3)}$.

Proof. If x' and x'' are elements of $G^{(3)}$, then the image by $\mu^{(3)}$ of the product $\mathrm{conc}_{G^{(3)}}(x'\otimes\big(x_1^{(1)}\otimes[y_1^{(1)},x_1^{(2)}]\otimes x_1^{(2)}+[x_1^{(1)},x_1^{(3)}]\otimes y_1^{(1)}\otimes y_1^{(3)}+x_1^{(3)}\otimes x_1^{(2)}\otimes[x_1^{(2)},y_1^{(3)}]\big)\otimes x'')$ is zero by Lemma B.2. $\square$

B.6. The maps $\mu_{Lie}^{p,q,r}$.

If p and q are integers > 0 and α (resp., β) is an integer in $\{1,\dots,p\}$ (resp., in $\{1,\dots,q\}$), we will denote by $\widetilde{M}_{pq}^{\alpha 0}$ (resp., $\widetilde{M}_{pq}^{0\beta}$) the set of all triples (c,c',ω) consisting of a pair (c,c') in $M_{pq}^{\alpha 0}$ (resp., in $M_{pq}^{0\beta}$) and of order relations on the sets $c^{-1}(1),\dots,c^{-1}(q),(c')^{-1}(1),\dots,(c')^{-1}(p)$.

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Proposition B.3. *Let (p,q) be a pair of integers > 0 and let $\alpha \in \{1, \ldots, p\}$ and $\beta \in \{1, \ldots, q\}$. Then there exist families of linear maps $(\Phi'(c, c', \omega))_{(c,c',\omega)\in\widetilde{M}^{\alpha 0}_{pq}}$ and $(\Phi''(c, c', \omega))_{(c,c',\omega)\in\widetilde{M}^{0\beta}_{pq}}$, indexed by $\widetilde{M}^{\alpha 0}_{pq}$ and $\widetilde{M}^{0\beta}_{pq}$ respectively, such that $\Phi'(c, c', \omega)$ maps FL_{p+q} to $FL_{p-\alpha}$ and $\Phi'(c, c', \omega)$ maps FL_{p+q} to $FL_{q-\beta}$, and such that the following identities are satisfied*

$$P(x_1^{(2)}, \ldots, x_q^{(2)}, x_1^{(3)}, \ldots, x_r^{(3)}) \otimes Q(x_1^{(1)}, \ldots, x_p^{(1)}, y_1^{(2)}, \ldots, y_q^{(2)}) \tag{45}$$
$$\otimes R(y_1^{(1)}, \ldots, y_p^{(1)}, y_1^{(3)}, \ldots, y_r^{(3)})$$

$$= \sum_{\alpha=1}^{p} \sum_{(c,c',\omega)\in\widetilde{M}^{\alpha 0}_{pq}} P\Big(\prod_{k_1\in(c')^{-1}(1)}\mathrm{ad}'(x_{k_1}^{(1)})(x_1^{(2)}), \ldots, \prod_{k_q\in(c')^{-1}(q)}\mathrm{ad}'(x_{k_q}^{(1)})(x_q^{(2)}), x_1^{(3)}, \ldots, x_r^{(3)}\Big)$$

$$\otimes \Phi'(c, c', \omega)(Q)(x_j^{(1)}, j \in (c')^{-1}(\infty))$$

$$\otimes R\Big(\prod_{l_1\in c^{-1}(1)}\mathrm{ad}'(y_{l_1}^{(2)})(y_1^{(1)}), \ldots, \prod_{l_p\in c^{-1}(p)}\mathrm{ad}'(y_{l_p}^{(2)})(y_p^{(1)}), y_1^{(3)}, \ldots, y_r^{(3)}\Big)$$

$$+ \sum_{\beta=1}^{q} \sum_{(c,c')\in\widetilde{M}^{0\beta}_{pq}} P\Big(\prod_{k_1\in(c')^{-1}(1)}\mathrm{ad}'(x_{k_1}^{(1)})(x_1^{(2)}), \ldots, \prod_{k_q\in(c')^{-1}(q)}\mathrm{ad}'(x_{k_q}^{(1)})(x_q^{(2)}), x_1^{(3)}, \ldots, x_r^{(3)}\Big)$$

$$\otimes \Phi''(c, c', \omega)(Q)(y_k^{(2)}, k \in c^{-1}(\infty))$$

$$\otimes R\Big(\prod_{l_1\in c^{-1}(1)}\mathrm{ad}'(y_{l_1}^{(2)})(y_1^{(1)}), \ldots, \prod_{l_p\in c^{-1}(p)}\mathrm{ad}'(y_{l_p}^{(2)})(y_p^{(1)}), y_1^{(3)}, \ldots, y_r^{(3)}\Big)$$

for any integer $r \geq 0$ and any pair of elements $(P, R) \in FA_{q+r} \times FA_{p+r}$.

Proof. Let us construct the maps $\Phi(c, c', \omega)$ and $\Phi''(c, c', \omega)$ by induction on $p + q$. Assume that these maps are constructed for all pairs of integers (p', q') with $p' + q' < p + q$. Let $(Q_\alpha)_\alpha$ be a basis of FL_{p+q}. We may assume that each Q_α is of the form $[Q'_\alpha, x_i^{(1)}]$, where $i \in \{1, \ldots, p\}$, or $[Q'_\alpha, y_j^{(2)}]$, where $j \in \{1, \ldots, q\}$, and $Q'_\alpha \in FL_{p+q-1}$. Let us treat the latter case. Then

$$P(x_1^{(2)}, \ldots, x_q^{(2)}, x_1^{(3)}, \ldots, x_r^{(3)}) \otimes [Q'_\alpha(x_1^{(1)}, \ldots, x_p^{(1)}, y_1^{(2)}, \ldots, \check{y}_j^{(2)}, \ldots, y_q^{(2)}), y_j^{(2)}]$$
$$\otimes R(y_1^{(1)}, \ldots, y_p^{(1)}, y_1^{(3)}, \ldots, y_r^{(3)})$$

may be rewritten as follows

$$\sum_{\alpha=1}^{p} \sum_{(c,c',\omega)\in\widetilde{M}_{p,q-1}^{\alpha0}} P(\prod_{k_1\in(c')^{-1}(1)} \mathrm{ad}'(x_{k_1}^{(1)})(x_1^{(2)}),\dots, \prod_{k_q\in(c')^{-1}(q)} \mathrm{ad}'(x_{k_q}^{(1)})(x_q^{(2)}), x_1^{(3)},\dots,x_r^{(3)})$$

$$\otimes\, [\Phi'(c,c',\omega)(Q'_\alpha)(x_j^{(1)}, j\in(c')^{-1}(\infty)), y_j^{(2)}]$$

$$\otimes\, R(\prod_{l_1\in c^{-1}(1)} \mathrm{ad}'(y_{l_1}^{(2)})(y_1^{(1)}),\dots, \prod_{l_p\in c^{-1}(p)} \mathrm{ad}'(y_{l_p}^{(2)})(y_p^{(1)}), y_1^{(3)},\dots,y_r^{(3)})$$

$$+\sum_{\beta=1}^{q-1} \sum_{(c,c',\omega)\in\widetilde{M}_{p,q-1}^{0\beta}} P(\prod_{k_1\in(c')^{-1}(1)} \mathrm{ad}'(x_{k_1}^{(1)})(x_1^{(2)}),\dots, \prod_{k_q\in(c')^{-1}(q)} \mathrm{ad}'(x_{k_q}^{(1)})(x_q^{(2)}), x_1^{(3)},\dots,x_r^{(3)})$$

$$\otimes\, [\Phi''(c,c',\omega)(Q'_\alpha)(y_k^{(2)}, k\in c^{-1}(\infty)), y_j^{(2)}]$$

$$\otimes\, R(\prod_{l_1\in c^{-1}(1)} \mathrm{ad}'(y_{l_1}^{(2)})(y_1^{(1)}),\dots, \prod_{l_p\in c^{-1}(p)} \mathrm{ad}'(y_{l_p}^{(2)})(y_p^{(1)}), y_1^{(3)},\dots,y_r^{(3)}),$$

applying the identity (45) to $P\otimes Q'_\alpha\otimes Ry_j^{(2)}$ and transferring the extreme right term $y_j^{(2)}$ to the middle tensor factor via the adjoint action. The second sum is of the desired form; we transform the first sum writing that

$$[\Phi'(c,c',\omega)(Q'_\alpha)(x_{j'}^{(1)}, j'\in(c')^{-1}(\infty)), y_j^{(2)}]$$
$$= \sum_{j'\in(c')^{-1}(\infty)} \Phi'(c,c',\omega)(Q'_\alpha)([x_{j''}^{(1)}, x_{j'}^{(1)}, y_j^{(2)}], j''\in(c')^{-1}(\infty)-\{j'\}).$$

and using the CYBE identity of Lemma B.3 to lower the degree of the middle term. This procedure and the condition that (45) holds defines uniquely the $\Phi'(c,c',\omega)(Q_\alpha)$ and $\Phi''(c,c',\omega)(Q_\alpha)$. Of course, the maps $\Phi'(c,c',\omega)$ and $\Phi''(c,c',\omega)$ are far from unique because of the many possible systems of bases of the FL_{p+q}. $\qquad\square$

Recall that we defined

$$F^{(aab)} = \oplus_{p+q\geq 0}(FL_p\otimes FL_q\otimes FL_{p+q})_{\mathfrak{S}_p\times\mathfrak{S}_q}, \quad F^{(abb)} = \oplus_{p+q\geq 0}(FL_n\otimes FL_p\otimes FL_q)_{\mathfrak{S}_p\times\mathfrak{S}_q}$$

Lemma B.4. *The natural inclusions of FL_k in FA_k ($k=p,q,p+q$) induce an inclusion of $F^{(aab)}\oplus F^{(abb)}$ in $F^{(3)}$.*

Proof. Each FL_k is a $\mathfrak{S}_k$-submodule of FA_k, therefore $FL_p\otimes FL_q\otimes FL_{p+q}$ is a $\mathfrak{S}_p\times\mathfrak{S}_q$-submodule of $FA_p\otimes FA_q\otimes FA_{p+q}$. The result now follows from the fact that if Γ is any finite group and $N\subset M$ is an inclusion of Γ-modules, then the natural map $N_\Gamma\to M_\Gamma$ is injective. (This fact is proven as follows: the representations of Γ are completely reducible, so M_Γ identifies with the multiplicity space of the trivial representation. Then $N\subset M$ corresponds to embeddings of the multiplicity spaces of each simple Γ-module, which shows that $N_\Gamma\subset M_\Gamma$.) $\qquad\square$

Corollary B.1. *Let p, q, r be integers > 0 and let P, Q, R be elements of FL_{q+r}, FL_{p+q} and FL_{p+r}. Then the element of $F^{(3)}$ equal to*

$$P(x_1^{(2)}, \ldots, x_q^{(2)}, x_1^{(3)}, \ldots, x_r^{(3)}) \otimes Q(x_1^{(1)}, \ldots, x_p^{(1)}, y_1^{(2)}, \ldots, y_q^{(2)})$$
$$\otimes R(y_1^{(1)}, \ldots, y_p^{(1)}, y_1^{(3)}, \ldots, y_r^{(3)})$$

of $\mu(G_{pqr})$ belongs to $F^{(aab)} \oplus F^{(abb)}$.

Proof. If P and R are Lie polynomials, then the expressions in the first and third tensor factors of the right side of (45) are also Lie polynomials. $\square$

$\mu_{Lie}^{p,q,r}(P \otimes Q \otimes R)$ is then defined as this element. $\square$

APPENDIX C. CONSTRUCTION AND PROPERTIES OF $\delta_4^{(F)}$ (PROOF OF PROPS. 3.3 AND 3.4)

In this Section, we first generalize the construction and the properties of the algebra $F^{(3)}$ to integers $n \geq 2$. The statements on $\delta_4^{(F)}$ will be immediate corollaries of these constructions.

C.1. Definition of $F^{(n)}$.

If n is an integer ≥ 2, let us define

$$P_n = \{\underline{p} = (p_{ij})_{1 \leq i < j \leq n} | p_{ij} \geq 0\}.$$

When $\underline{p}$ belongs to P_n, let us set

$$F_{\underline{p}}^{(n)} = \big(\bigotimes_{i=1}^{n} (FA_{\sum_{j|j<i} p_{ji}} \otimes FA_{\sum_{j|j>i} p_{ij}}) \big)_{\prod_{1 \leq i < j \leq n} \mathfrak{S}_{p_{ij}}}.$$

Here the generators of the first part of the ith factor are denoted $x_\alpha^{(ji)}$, where (j, α) are such that $j < i$ and $1 \leq \alpha \leq p_{ji}$, and the generators of the second part of the ith factor are denoted $y_\alpha^{(ij)}$, where (j, α) are such that $j > i$ and $1 \leq \alpha \leq p_{ij}$. There are therefore $n(n-1)$ sets of generators, indexed by the pairs (i, j) such that $(i, j) \in \{1, \ldots, n\}^2$ and $i \neq j$. Let us put $S_{ij} = \{y_\alpha^{(ij)} | \alpha = 1, \ldots, p_{ij}\}$ when $i < j$ and $S_{ij} = \{x_\alpha^{(ji)} | \alpha = 1, \ldots, p_{ji}\}$ when $i > j$. If $i < j$, there is a bijection between S_{ij} and S_{ji}, sending each $y_\alpha^{(ij)}$ to $x_\alpha^{(ij)}$. Then $\mathfrak{S}_{p_{ij}}$ acts by simultaneaous permutation of the sets S_{ij} and S_{ji} and therefore also on $\bigotimes_{i=1}^{n} (FA_{\sum_{j|j<i} p_{ji}} \otimes FA_{\sum_{j|j>i} p_{ij}})$. $F_{\underline{p}}^{(n)}$ is then defined as the space of coinvariants of this action. We set

$$F^{(n)} = \bigoplus_{\underline{p} \in P_n} F_{\underline{p}}^{(n)}.$$

C.2. Basis of $F^{(n)}$.

When $\underline{s} \in \mathbb{N}^n$, let us set $I_{\underline{s}} = \{(i, \alpha) | 1 \leq i \leq n, 1 \leq \alpha \leq s_i\}$. Let us denote by ind the map from $I_{\underline{s}}$ to $\{1, \ldots, n\}$ such that $\text{ind}(i, \alpha) = i$. We set $I_{\underline{s},i} = \text{ind}^{-1}(\{i\})$.

Let us define Φ_n as the set of all triples $(\underline{s}, \underline{t}, \phi)$, where $\underline{s}$ and $\underline{t}$ belong to $\mathbb{N}^n$ and ϕ is a bijection from $I_{\underline{s}}$ to $I_{\underline{t}}$, such that for any $(i, \alpha) \in I_{\underline{s}}$, $\text{ind}(\phi(i, \alpha)) > i$. (If $(\underline{s}, \underline{t}, \phi) \in \Phi_n$, we have therefore $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i$ and $s_1 = t_n = 0$.)

There is a unique map π_n from Φ_n to P_n, such that for any $(\underline{s}, \underline{t}, \phi)$ in Φ_n, $\pi_n(\underline{s}, \underline{t}, \phi)$ is the element $\underline{p} = (p_{ij})_{1 \leq i < j \leq n}$ of P_n such that $p_{ij} = \text{card}(\phi(I_{\underline{s},i}) \cap I_{\underline{t},j})$. In particular, if $\underline{p} = \pi_n(\underline{s}, \underline{t}, \phi)$, then we have $s_i = \sum_{j|j>i} p_{ij}$ and $t_i = \sum_{j|j<i} p_{ji}$.

If $(\underline{s}, \underline{t}, \phi) \in \Phi_n$, let us set

$$A_{ij} = I_{\underline{s},i} \cap \phi^{-1}(I_{\underline{t},j}), \quad B_{ij} = \phi(I_{\underline{s},i}) \cap I_{\underline{t},j}.$$

Then A_{ij} and B_{ij}) are empty when $j \leq i$; $(A_{i\alpha})_{\alpha|\alpha>i}$ is a partition of $I_{\underline{s},i}$ and $(B_{\alpha i})_{\alpha|\alpha<i}$ is a partition of $I_{\underline{t},i}$.

If $(\underline{s}, \underline{t}, \phi) \in P_n$, define $z(\underline{s}, \underline{t}, \phi)$ in $F^{(n)}_{\pi_n(\underline{s},\underline{t},\phi)}$ as

$$z(\underline{s}, \underline{t}, \phi) = \bigotimes_{i=1}^{n} (z_1^{(i)} \cdots z_{s_i}^{(i)}) \otimes (w_1^{(i)} \cdots w_{t_i}^{(i)}),$$

where if $(a_{ij,\alpha})_{\alpha=1,\dots,p_{ij}}$ is the increasing sequence such that $A_{ij} = \{a_{ij,1}, \dots, a_{ij,p_{ij}}\}$, then

$$z_{a_{ij,\alpha}}^{(i)} = x_{\alpha}^{(ij)} \quad \text{and} \quad w_{\phi(a_{ij,\alpha})}^{(j)} = y_{\alpha}^{(ij)}.$$

Lemma C.1. *The family* $(z(\underline{s}, \underline{t}, \phi))_{(\underline{s},\underline{t},\phi)\in P_n}$ *is a basis of* $F^{(n)}$.

The families $(z_{\alpha}^{(i)})$ and $(w_{\beta}^{(i)})$ may also be defined as follows. Let $(b_{ij,\alpha})_{\alpha=1,\dots,p_{ij}}$ be the increasing sequence such that $B_{ij} = \{b_{ij,1}, \dots, b_{ij,p_{ij}}\}$, then

$$z_{\phi^{-1}(b_{ij,\alpha})}^{(i)} = x_{ij}^{(\alpha)} \quad \text{and} \quad w_{b_{ij,\alpha}}^{(j)} = y_{\alpha}^{(ij)}.$$

C.3. **Product in $F^{(n)}$.** If $(\underline{s}, \underline{t}, \phi)$ and $(\underline{s}', \underline{t}', \phi')$ are elements of P_n, and if for each $\alpha = 1, \dots, n$, (c_α, c'_α) is an element of M_{t_α, s'_α} let us define $\underline{p}((\underline{s}, \underline{t}, \phi), (\underline{s}', \underline{t}', \phi'), (c_\alpha, c'_\alpha)_{\alpha=1,\dots,n})$ as the element of P_n such that if $1 \le i < j \le n$, then

$$
\underline{p}((\underline{s}, \underline{t}, \phi), (\underline{s}', \underline{t}', \phi'), (c_\alpha, c'_\alpha)_{\alpha=1,\dots,n})
$$
$$
= \mathrm{card}(c_i^{-1}(-\infty) \cap A'_{ij}) + \mathrm{card}(c_j'^{-1}(\infty) \cap B_{ij})
$$
$$
+ \sum_{k=i+1}^{j-1} \mathrm{card}(c_k'^{-1}(A'_{kj}) \cap B_{ik}) + \mathrm{card}(c_k^{-1}(B_{ik}) \cap A'_{kj})
$$

where $(A'_{ij})_{j|j>i}$ and $(B'_{ij})_{j|j<i}$ denote the partitions of $I_{\underline{s}',i}$ and $I_{\underline{t}',i}$ associated to $(\underline{s}', \underline{t}', \phi')$. Let us define now $z((\underline{s}, \underline{t}, \phi), (\underline{s}', \underline{t}', \phi'), (c_\alpha, c'_\alpha)_{\alpha=1,\dots,n})$ as the element of $F^{(n)}_{\underline{p}((\underline{s},\underline{t},\phi),(\underline{s}',\underline{t}',\phi'),(c_1,c'_1),\dots,(c_n,c'_n))}$ equal to

$$z((\underline{s}, \underline{t}, \phi), (\underline{s}', \underline{t}', \phi'), (c_\alpha, c'_\alpha)_{\alpha=1,\dots,n}) \tag{46}$$

$$
= \bigotimes_{i=1}^{n} \Big(\overset{\nearrow}{\prod_{\alpha \in c_{j_{i,1}}^{-1}(\phi(i,1))}} \mathrm{ad}'(z_\alpha'^{(j_{i,1})})(z_1^{(i)}) \cdots \overset{\nearrow}{\prod_{\alpha \in c_{j_{i,s_i}}^{-1}(\phi(i,s_i))}} \mathrm{ad}'(z_\alpha'^{(j_{i,s_i})})(z_{s_i}^{(i)}) \prod_{\alpha \in c_i^{-1}(-\infty)} z_\alpha'^{(i)} \Big)
$$

$$
\otimes \Big(\overset{\nearrow}{\prod_{\alpha \in c_i'^{-1}(\infty)}} w_\alpha^{(i)} \overset{\searrow}{\prod_{\alpha \in c_{k_{i,1}}'^{-1}(\phi^{-1}(i,1))}} \mathrm{ad}'(w_\alpha^{(k_{i,1})})(w_1'^{(i)}) \cdots \overset{\searrow}{\prod_{\alpha \in c_{k_{i,t_i}}'^{-1}(\phi^{-1}(i,t_i))}} \mathrm{ad}'(w_\alpha^{(k_{i,t_i})})(w_{t_i}'^{(i)}) \Big),
$$

where we set $j_{i,1} = \mathrm{ind}(\phi(i,1)), \dots, j_{i,s_i} = \mathrm{ind}(\phi(i,s_i))$ and $k_{i,1} = \mathrm{ind}(\phi^{-1}(i,1)), \dots, k_{i,t_i} = \mathrm{ind}(\phi^{-1}(i,t_i))$.

In the right side of (46), the sets of generators of $F^{(n)}_{\underline{p}((\underline{s},\underline{t},\phi),(\underline{s}',\underline{t}',\phi'),(c_1,c'_1),\dots,(c_n,c'_n))}$ are the following:

– if $i < j$, S_{ij} consists of the $z'^{(i)}_\alpha$, $\alpha \in c_i^{-1}(-\infty) \cap A'_{ij}$, the $z^{(i)}_\beta$, where (i, β) belongs to $\phi^{-1}(c_j'^{-1}(\infty) \cap B_{ij})$, the $z'^{(k)}_\gamma$, where $\gamma \in c_k^{-1}(B_{ik}) \cap A'_{kj}$ and the $z^{(i)}_\delta$, where $(i, \delta) \in \phi^{-1}\big((c_k')^{-1}(A'_{kj} \cap B_{ik})\big)$

– if $i < j$, S_{ji} consists of the $w'^{(j)}_\beta$, where $\beta \in \phi(c_i^{-1}(\infty) \cap A'_{ij})$, the $w^{(j)}_\alpha$, where $\alpha \in (c_j')^{-1}(\infty) \cap B_{ij}$, the $w'^{(j)}_\delta$, where $(j, \delta) \in \phi\big(c_k^{-1}(B_{ik}) \cap A'_{kj}\big)$, and the $w^{(k)}_\gamma$, where $\gamma \in (c_k')^{-1}(A'_{kj}) \cap B_{ik}$.

The bijection from S_{ij} to S_{ji} maps each $z'^{(i)}_\alpha$ to $w'^{(j)}_\beta$, where $(j, \beta) = \phi(i, \alpha)$, each $z^{(i)}_\beta$ to $w^{(j)}_\alpha$, where $(j, \alpha) = \phi(i, \beta)$, each $z'^{(k)}_\gamma$ to $w'^{(j)}_\delta$, where $(j, \delta) = \phi(k, \gamma)$ and each $z^{(i)}_\delta$ to $w^{(k)}_\gamma$, where $(k, \gamma) = \phi(i, \delta)$.

Define $m_{F^{(n)}}$ as the unique linear map from $F^{(n)} \otimes F^{(n)}$ to $F^{(n)}$, such that for any $(\underline{s}, \underline{t}, \phi)$ and $(\underline{s}', \underline{t}', \phi')$ in P_n, we have

$$m_{F^{(n)}}(z(\underline{s}, \underline{t}, \phi) \otimes z(\underline{s}', \underline{t}', \phi')) = \sum_{(c_1, c_1') \in M_{t_1, s_1'}, \dots, (c_n, c_n') \in M_{t_n, s_n'}}$$

A construction analogous to that of Appendix B shows

Proposition C.1. $(F^{(n)}, m_{F^{(n)}})$ *is an associative algebra.*

Remark 12. When $n = 3$, the connection between this presentation of $F^{(3)}$ and that of Appendix B relies on the identification $(p_{12}, p_{13}, p_{23}) = (q, r, p)$.

On the other hand, $F^{(2)}$ coincides with the direct sum $\oplus_{n \geq 0} \mathbb{K}\mathfrak{S}_n$, where the product is the linear extension of the concatenation of permutations.

C.4. **Universal property of $F^{(n)}$.** Let $(\mathfrak{a}, r_\mathfrak{a})$ be the pair of a Lie algebra $\mathfrak{a}$ and a solution $r_\mathfrak{a} = \sum_{i \in I} a_i \otimes b_i \in A \otimes A$ of CYBE. Then there is a unique map

$$\kappa^{(n)}_{\mathfrak{a}, r_\mathfrak{a}} : F^{(n)} \to T\mathfrak{a} \otimes U\mathfrak{a} \otimes T\mathfrak{a}$$

such that for any $(\underline{s}, \underline{t}, \phi) \in P_n$,

$$\kappa^{(n)}_{\mathfrak{a}, r_\mathfrak{a}}(z(\underline{s}, \underline{t}, \phi)) = \sum_{\alpha \in \mathrm{Map}(I_{\underline{s}}, I)} \bigotimes_{i=1}^n \big(\prod_{\gamma \in I_{\underline{s}, i}} a(\alpha(i, \gamma)) \prod_{\delta \in I_{\underline{t}, i}} b(\alpha(\phi^{-1}(i, \delta))) \big);$$

we denote by $\mathrm{Map}(I_{\underline{s}}, I)$ the set of all maps from $I_{\underline{s}}$ to I.

Proposition C.2. $\kappa^{(n)}_{\mathfrak{a}, r_\mathfrak{a}}$ *is a morphism of algebras.*

C.5. **The normal ordering map.** When $\underline{p} \in P_n$, let us set

$$G^{(n)}_{\underline{p}} = \big(\bigotimes_{i=1}^n FA_{\sum_{j|j<i} p_{ji} + \sum_{j|j>i} p_{ij}} \big)_{\prod_{1 \leq i < j \leq n} \mathfrak{S}_{p_{ij}}}.$$

The generators of the ith factor of the tensor product are $u^{(i)}_1, \dots, u^{(i)}_{\sum_{j|j<i} p_{ji} + \sum_{j|j>i} p_{ij}}$. Let us denote by S_i this set of generators. We have a partition $S_i = \cup_{j|j \neq i} S_{ij}$, where

– if $j < i$, S_{ij} consists of the $u^{(i)}_{\sum_{j'|j'<j} p_{j'i}+1}, \ldots, u^{(i)}_{\sum_{j'|j'\leq j} p_{j'i}}$

– if $j > i$, S_{ij} consists of the $u^{(i)}_{\sum_{j'|j'<i} p_{j'i}+\sum_{j'|i<j'<j} p_{ij'}+1}, \ldots, u^{(i)}_{\sum_{j'|j'<i} p_{j'i}+\sum_{j'|i<j'\leq j} p_{ij'}}$.

When $i < j$, there is a bijection from S_{ij} to S_{ji}, sending each $u^{(i)}_{\sum_{j'|j'<i} p_{j'i}+\sum_{j'|i<j'<j} p_{ij'}+\alpha}$ to $u^{(j)}_{\sum_{k'|k'<j} p_{k'j}+\alpha}$, for $\alpha = 1, \ldots, p_{ij}$. Then $\mathfrak{S}_{p_{ij}}$ acts by simultaneously permuting the elements of S_{ij} and of S_{ji}.

When $(\sigma_i)_{i=1,\ldots,n}$ is a collection of permutations of $\prod_{i=1}^n \mathfrak{S}_{\sum_{j|j<i} p_{ji}+\sum_{j|j>i} p_{ij}}$, let us set

$$u\big((\sigma_i)_{i=1,\ldots,n}\big) = \bigotimes_{i=1}^n \big(u^{(i)}_{\sigma_i(1)} \cdots u^{(i)}_{\sum_{j|j<i} p_{ji}+\sum_{j|j>i} p_{ij}}\big).$$

Then for any $\underline{p} \in P_n$, the family $\{u\big((\sigma_i)_{i=1,\ldots,n}\big)|(\sigma_i)_{i=1,\ldots,n} \in \prod_{i=1}^n \mathfrak{S}_{\sum_{j|j<i} p_{ji}+\sum_{j|j>i} p_{ij}}\}$ is a generating family of $G_{\underline{p}}^{(n)}$.

When $\underline{p} \in P_n$, let us set $\mathfrak{S}(\underline{p}) = \prod_{i=1}^n \mathfrak{S}_{\sum_{j|j<i} p_{ji}+\sum_{j|j>i} p_{ij}}$. If $\underline{p}$ and $\underline{q}$ belong to P_n and $\underline{\sigma} = (\sigma_i)_{i=1,\ldots,n}$ (resp., $\underline{\tau} = (\tau_i)_{i=1,\ldots,n}$) belongs to $\mathfrak{S}(\underline{p})$ (resp., to $\mathfrak{S}(\underline{q})$), define $\underline{\sigma} * \underline{\tau}$ as the element of $\mathfrak{S}(\underline{p}+\underline{q})$ such that for any $i = 1, \ldots, n$, $((\underline{\sigma} * \underline{\tau})_i)(\alpha) = \sigma_i(\alpha)$ if $\alpha \leq \sum_{j|j>i} p_{ij} + \sum_{j|j<i} p_{ji}$, and $((\underline{\sigma} * \underline{\tau})_i)(\alpha) = \tau_i(\alpha - (\sum_{j|j>i} p_{ij} + \sum_{j|j<i} p_{ji})) + \sum_{j|j>i} p_{ij} + \sum_{j|j<i} p_{ji}$ if $\alpha \leq \sum_{j|j>i} p_{ij} + \sum_{j|j<i} p_{ji}$.

Then there is a unique linear map $\mathrm{conc}_{G^{(n)}}$ from $G^{(n)} \otimes G^{(n)}$ to $G^{(n)}$, such that if $\underline{p}, \underline{q} \in P_n$ and $\underline{\sigma} \in \mathfrak{S}(\underline{p})$, $\underline{\tau} \in \mathfrak{S}(\underline{q})$, then

$$\mathrm{conc}_{G^{(n)}}(u(\underline{\sigma}) \otimes u(\underline{\tau})) = u(\underline{\sigma} * \underline{\tau}).$$

When $k = 1, \ldots, n-1$, define $G^{k,(n)}$ as the subspace of $G^{(n)}$ spanned by all $u(\underline{\sigma})$, where $\underline{\sigma}$ is such that when $k' \leq k$, the generators of $\cup_{k'|k'<k} S_{kk'}$ occur before the generators of $\cup_{k'|k'>k} S_{k'k}$ in the kth factor. Then $F^{(n)}$ and $G^{(n)}$ may be identified with $G^{n-1,(n)}$ and $G^{1,(n)}$. When $k = 1, \ldots, n-2$, define μ_k as the following partial ordering map. μ_k maps $G^{k,(n)}$ to $G^{k+1,(n)}$. If $\underline{p} \in P_n$, set $p(i,j) = p_{ij}$ if $i < j$ and $p(i,j) = p_{ji}$ if $i > j$. Then if $\underline{\sigma} \in \mathfrak{S}(\underline{p})$, then one may find elements of free algebras P_j, $j \in \{1, \ldots, n\} - \{k\}$ and partitions $\{1, \ldots, p(k,k')\} = \cup_{r=1}^s I_{rk'}$, such that $u(\underline{\sigma})$ has the form

$$\bigotimes_{j|j<k} P_j(u_\alpha^{(jj')}|j' \neq j, \alpha = 1, \ldots, p(j,j')) \otimes$$

$$\prod_{r=1}^s \big(\prod_{i|i\neq k}\big(\prod_{\alpha\in I_{ri}} u_\alpha^{(ki)}\big)\big) \otimes \bigotimes_{j|j>k} P_j(u_\alpha^{(jj')}|j' \neq j, \alpha = 1, \ldots, p(j,j')).$$

Let us set $q_t = \sum_{t'<k} \mathrm{card}(I_{tt'})$ and $p_t = \sum_{t'>k} \mathrm{card}(I_{tt'})$ for $t = 1, \ldots, s$. When $u < t$ and $\alpha = 1, \ldots, \mathrm{card}(I_{tu})$, let us set $w'_{q_1+\cdots+q_{t-1}+\mathrm{card}(I_{t1})+\cdots+\mathrm{card}(I_{t,u-1})+\alpha} = u^{(ku)}_{n_\alpha}$, where n_α is the αth element of I_{tu}. When $u > t$ and $\alpha = 1, \ldots, \mathrm{card}(I_{tu})$,

let us set $v_{p_1+\cdots+p_{t-1}+\mathrm{card}(I_{t,k+1})+\cdots+\mathrm{card}(I_{t,u-1})+\alpha} = u_{n_\alpha}^{(ku)}$. Let us also set $v'_p = u_\nu^{(uk)}$ whenever $w'_p = u_\nu^{(ku)}$ and $w_p = u_\nu^{(ku)}$ whenever $v_p = u_\nu^{(uk)}$.

Let us set $A_i = \cup_{t=1}^s (q_1+\cdots+q_{t-1}+\mathrm{card}(I_{t1})+\cdots+\mathrm{card}(I_{t,i-1})+\{1,\ldots,\mathrm{card}(I_{ti})\})$ and $B_j = \cup_{t=1}^s (p_1+\cdots+p_{t-1}+\mathrm{card}(I_{t,k+1})+\cdots+\mathrm{card}(I_{t,j-1})+\{1,\ldots,\mathrm{card}(I_{tj})\})$.

Let us set $\underline{p} = (p_1,\ldots,p_s)$ and $\underline{p} = (q_1,\ldots,q_s)$. For $(c,c') \in M_{\underline{q},\underline{p}}$, let us define $(p_{ij}(c,c'))_{1\le i<j\le n}$ as follows. If $i < j < k$ or $k < i < j$, then $p_{ij}(c,c') = p_{ij}$. If $i < k < j$, then $p_{ij}(c,c') = p_{ij} + \mathrm{card}(A_j \cap (c')^{-1}(B_i))$. If $i < k$, then $p_{ik} = \mathrm{card}(c^{-1}(-\infty) \cap A_i)$. If $k < j$, then $p_{kj} = \mathrm{card}((c')^{-1}(\infty) \cap B_j)$.

When $(c,c') \in M_{\underline{q},\underline{p}}$, define $u(k,\underline{\sigma}),c,c'$ as the element of $G^{(n)}_{(p_{ij}(c,c'))_{1\le i<j\le n}}$ equal to

$$\bigotimes_{j|j<k} P_j(u_\alpha^{(jj')}|j'\ne j,k; \alpha = 1,\ldots,p(j,j'); \prod_{j\in(c')^{-1}(i')}^{\searrow} \mathrm{ad}'(v_j)(v'_{i'}), i' \in A_j)$$

$$\otimes \prod_{i'\in c^{-1}(-\infty)}^{\nearrow} w'_{i'} \prod_{i\in(c')^{-1}(\infty)}^{\nearrow} v_i$$

$$\otimes \bigotimes_{j|j>k} P_j(u_\alpha^{(jj')}|j'\ne j,k; \alpha = 1,\ldots,p(j,j'); \prod_{j'\in c^{-1}(i)}^{\nearrow} \mathrm{ad}'(w'_{j'})(w_i), i \in B_j))$$

Then there is a unique map $\mu_k : G^{k,(n)} \to G^{k+1,(n)}$ such that $\mu_k(u(\underline{\sigma})) = \sum_{(c,c')\in M_{\underline{q},\underline{p}}} u(k,\underline{\sigma}),c,c'$.

Define now the linear map $\mu^{(n)} : F^{(n)} \to G^{(n)}$ as the composition $\mu_{n-2}\circ\cdots\circ\mu_2$.

Then $\mu^{(n)}$ is an algebra morphism from $(F^{(n)}, m_{F^{(n)}})$ to $(G^{(n)}, \mathrm{conc}_{G^{(n)}})$. $\mu^{(n)}$ is a normal ordering map is a sense generalizing Proposition B.2.

Moreover, normal ordering commutes with the operation of "mixing together" elements of $G^{(n)}$. If N is an integer and $(N_1,\ldots,N_n)$ is a partition of N, let us define $G^{(N)}(N_1,\ldots,N_n)$ as the direct sum of all $G_{\underline{p}}^{(N)}$, where $\underline{p}$ is such that for any $k = 1,\ldots,n$ and any pair i,j of elements of $N_1+\cdots+N_{k-1}+\{1,\ldots,N_k\}$, $p_{ij} = 0$. Assume that $(N^{(\beta)})_{\beta=1,\ldots,\alpha}$ is a family of integers and for each $\beta = 1,\ldots,\alpha$, $(N_\alpha^{(\beta)})$ is a partition of $N^{(\beta)}$. Assume that for $\beta = 1,\ldots,\alpha$, $x_\beta \in G^{(N^{(\beta)})}(N_1^{(\beta)},\ldots,N_n^{(\beta)})$, and let us fix a sequence of bijections $b_k : \{1,\ldots,\sum_\beta N_k^{(\beta)}\} \to I_{N_k^{(1)},\ldots,N_k^{(\beta)}}$. To the sequence $\underline{b} = (b_k)_{k=1,\ldots,n}$, we associate a map

$$\mathrm{conc}_{\underline{b}} : \bigotimes_{\beta=1}^\alpha G^{(N^{(\beta)})}(N_1^{(\beta)},\ldots,N_N^{(\beta)}) \to G^{(n)},$$

such that if (b'_k, b''_k) are the components of the map b_k, then the first factor of $\mathrm{conc}_{\underline{b}}(\otimes_{\beta=1}^\alpha x_\beta)$ is the concatenation of the $b''_1(1)$th factor of $x_{b'(1)}$, the $b''_1(2)$th factor of $x_{b'(2)}$, etc, the ith factor of $\mathrm{conc}_{\underline{b}}(\otimes_{\beta=1}^\alpha x_\beta)$ is the concatenation of $N_1^{(b'(i))}+\cdots+$

$N_{i-1}^{(b'(i))} + b''_i(1)$th factor of $x_{b'(i)}$, the $N_1^{(b'(i))} + \cdots + N_{i-1}^{(b'(i))} + b''_i(2)$th factor of $x_{b'(i)}$, etc.

Then we have

Proposition C.3. $\mu^{(n)} \circ \mathrm{conc}_{\underline{b}} = \mu^{(n)} \circ \mathrm{conc}_{\underline{b}} \circ (\otimes_{\beta=1}^{\alpha} \mu^{(N^{(\beta)})})$.

Proof. We have seen that the image by $\mu^{(3)}$ of an element of $G^{(3)}$ is not changed if we apply the CYBE identity within in this element. Therefore the same is true for any map $\mu^{(k)}$. Now each map $\mu^{(N^{(\alpha)})}$ consists of repeated applications of the CYBE identity, so the image by $\mu^{(N)}$ of $\mathrm{conc}_{\underline{b}}(\otimes_{\beta=1}^{\alpha} x_\beta)$ and $\mathrm{conc}_{\underline{b}}(\otimes_{\beta=1}^{\alpha} \mu^{(N^{(\beta)})}(x_\beta))$ will be the same. $\square$

Using $\mu^{(n)}$, one may prove in the same way as Corollary B.1:

Proposition C.4. *Assume that for $i = 1, \ldots, n$, L_i is an element of $FL_{\sum_{j|j<i} p_{ji} + \sum_{j|j>i} p_{ij}}$. Then the element of $F^{(n)}$ equal to*

$$\mu^{(n)}\Big(\bigotimes_{i=1}^{n} L_i(u_1^{(i)}, \ldots, u_{\sum_{j|j<i} p_{ji} + \sum_{j|j>i} p_{ij}}^{(i)}) \Big)$$

belongs to $\bigoplus_{x_2,\ldots,x_{n-2}\in\{a,b\}} F^{(ax_2\cdots x_{n-2}b)}$.

As it happens in the case $n = 3$, the restriction on $\underline{p}$ (see equation (18)) comes from the fact that the reorderings due to CYBE tranfer the $x_\alpha^{(i)}$ to the left and the $y_\alpha^{(i)}$ to the right, so the fact that each $x_\alpha^{(i)}$ is at the left of the corresponding $y_\alpha^{(i)}$ is not changed.

C.6. Construction and properties of $\delta_4^{(F)}$.

C.6.1. *The maps $x \mapsto x^{(i_1 \ldots i_p)}$.* Let $i_1, \ldots, i_p$ be integers such that $1 \leq i_1 < \cdots < i_p \leq n$. Then there is a unique linear map $\underline{s} \mapsto \underline{s}^{(i_1 \ldots i_p)}$ from $\mathbb{N}^p$ to $\mathbb{N}^n$, such that $\underline{s}_i^{(i_1 \ldots i_p)} = 0$ if $i \notin \{i_1, \ldots, i_p\}$ and $\underline{s}_{i_j}^{(i_1 \ldots i_p)} = s_j$ for $j = 1, \ldots, p$. If $(\underline{s}, \underline{t}, \phi)$ belongs to Φ_p, define $\phi^{(i_1 \ldots i_p)}$ as the map from $I_{\underline{s}^{(i_1 \ldots i_p)}}$ to $I_{\underline{t}^{(i_1 \ldots i_p)}}$ such that if $(k, \alpha) \in I_{\underline{s}}$ and $(\phi(k, \alpha)) = (l, \beta)$, then $\phi^{(i_1 \ldots i_p)}(i_k, \alpha) = (i_l, \beta)$. Denote also by $x \mapsto x^{(i_1 \ldots i_p)}$ the map from Φ_p to Φ_n, such that $(\underline{s}, \underline{t}, \phi)^{(i_1 \ldots i_p)} = (\underline{s}^{(i_1 \ldots i_p)}, \underline{t}^{(i_1 \ldots i_p)}, \phi^{(i_1 \ldots i_p)})$. Let us also denote by $x \mapsto x^{(i_1 \ldots x_p)}$ the linear map from $F^{(p)}$ to $F^{(n)}$, such that for any $(\underline{s}, \underline{t}, \phi) \in \Phi_p$, $z(\underline{s}, \underline{t}, \phi)^{(i_1 \ldots i_p)} = z((\underline{s}, \underline{t}, \phi)^{(i_1 \ldots i_p)})$. Then $x \mapsto x^{(i_1 \ldots x_p)}$ is an algebra morphism from $(F^{(p)}, m_{F^{(p)}})$ to $(F^{(n)}, m_{F^{(n)}})$.

C.6.2. Let us denote by r the element of $F_1^{(2)}$ equal to $x_1^{(12)} \otimes y_1^{(12)}$.

Lemma C.2. *1) Let ρ belong to $\bigoplus_{x_1,\ldots,x_{n-3}\in\{a,b\}} F^{(ax_1\ldots x_{n-3}b)}$. Then if $i < j$ (resp., if $j < i$), the commutators $[r^{(ij)}, \rho^{1\ldots i-1,i+1\ldots n}]$ (resp., $[r^{(ji)}, \rho^{1\ldots i-1,i+1\ldots n}]$) belongs to $\bigoplus_{x_1,\ldots,x_{n-3}\in\{a,b\}} F^{(ax_1\ldots x_{n-2}b)}$.*

2) If $i < j < k$, we have $[r^{(ij)}, r^{(ik)}] + [r^{(ij)}, r^{(jk)}] + [r^{(ik)}, r^{(jk)}] = 0$ and if i, j, k, l are all distinct and $i < j$, and $k < l$, then $[r^{(ij)}, r^{(kl)}] = 0$.

Proof. It suffices to prove 1) when x belongs to some $F^{(ax_1\ldots x_{n-3}b)}$ and is the tensor product of Lie polynomials. There is nothing to do when $i < j$ and $x_{j-1} = b$, or when $i > j$ and $x_j = a$. In the two other cases, one applies Proposition C.4 to a suitable family of Lie polynomials.

The first part of 2) is a consequence of the fact that the map $x \mapsto x^{(ijk)}$ from $F^{(3)}$ to $F^{(n)}$ is an algebra morphism. The second part is immediate. □

When $x \in F^{(3)}$ define $\delta^{F^{(3)} \to F^{(4)}}(x)$ as follows

$$\delta^{F^{(3)} \to F^{(4)}}(x) = [r^{(12)} + r^{(13)} + r^{(14)}, x^{(234)}] - [-r^{(12)} + r^{(23)} + r^{(24)}, x^{(134)}]$$
$$+ [-r^{(13)} - r^{(23)} + r^{(34)}, x^{(124)}] - [-r^{(14)} - r^{(24)} - r^{(34)}, x^{(123)}].$$

$\delta^{F^{(3)} \to F^{(4)}}$ maps $F^{(3)}$ to $F^{(4)}$. It follows from Lemma C.2, 1) that if $x \in F^{(aab)} \oplus F^{(abb)}$, $\delta^{F^{(3)} \to F^{(4)}}$ maps maps $F^{(aab)} \oplus F^{(abb)}$ to $\oplus_{x,y \in \{a,b\}} F^{(axyb)}$. We define $\delta_4^{(F)}$ as the resulting map from $F^{(aab)} \oplus F^{(abb)}$ to $\oplus_{x,y \in \{a,b\}} F^{(axyb)}$.

Then it follows from Proposition C.2 that $\delta_4^{(F)}$ satisfies the conclusion of 3.3.

Recall that $\delta_3^{(F)}$ is the restriction to $\prod_{n \geq 0} F_n$ of the map $\delta^{F^{(2)} \to F^{(3)}} : F^{(2)} \to F^{(3)}$ such that

$$\delta^{F^{(2)} \to F^{(3)}}(x) = [r^{(12)}, x^{(13)}] + [r^{(12)}, x^{(23)}] + [r^{(13)}, x^{(23)}] + [x^{(12)}, r^{(13)}] + [x^{(12)}, r^{(23)}] + [x^{(13)}, r^{(23)}].$$

Then it follows from Lemma C.2, 2) that the composition $\delta^{F^{(3)} \to F^{(4)}} \circ \delta^{F^{(2)} \to F^{(3)}}$, which is a map from $F^{(2)}$ to $F^{(4)}$, is zero. It follows that the composition $\delta_4^{(F)} \circ \delta_3^{(F)}$ is also zero.

This proves Proposition 3.4, 2).

Remark 13. It is now clear how to define $\delta_n^{(F)} : \oplus_{x_1,\ldots,x_{n-3} \in \{a,b\}} F^{(ax_1 \cdots x_{n-3}b)} \to \oplus_{y_1,\ldots,y_{n-2} \in \{a,b\}} F^{(ay_1 \cdots y_{n-2}b)}$, such that $\delta_n^{(F)} \circ \delta_{n-1}^{(F)} = 0$.

APPENDIX D. COMPUTATION OF COHOMOLOGY GROUPS

D.1. **Computation of H_n^2.** Clearly, $\text{Ker}(\delta_3^{(F)}) \cap F_1 = \mathbb{K}r$. Let us assume now that $n > 1$. We want to prove that $\text{Ker}(\delta_3^{(F)}) \cap F_n = 0$.

Let x belong to $\text{Ker}(\delta_3^{(F)}) \cap F_n$. Since the family $(x_1^{(12)} \cdots x_n^{(12)} \otimes y_{\sigma(1)}^{(12)} \cdots y_{\sigma(n)}^{(12)})_{\sigma \in \mathfrak{S}_n}$ is a basis of $F_n^{(2)} = (FA_n \otimes FA_n)_{\mathfrak{S}_n}$, there exists a unique family $(A_\sigma)_{\sigma \in \mathfrak{S}_n}$ in $\mathbb{K}^{\mathfrak{S}_n}$ such that

$$x = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_1^{(12)} \cdots x_n^{(12)} \otimes y_{\sigma(1)}^{(12)} \cdots y_{\sigma(n)}^{(12)}.$$

Moreover:

Lemma D.1. *The condition that x belong to $(FL_n \otimes FL_n)_{\mathfrak{S}_n}$ is equivalent to the condition that $\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ and $\sum_{\sigma \in \mathfrak{S}_n} A_{\sigma^{-1}} x_{\sigma(1)} \cdots x_{\sigma(n)}$ are both Lie polynomials in the free algebra with generators $x_1, \ldots, x_n$.*

Let π_{01n} denote the projection of $F^{(3)}$ on F_{01n} parallel to $\bigoplus_{(p,q,r)|(p,q,r)\neq(0,1,n)} F_{pqr}$. Let us apply π_{01n} to the identity $\delta_3^{(F)}(x) = 0$.

Since $[x^{12}, r^{13}] \in F_{0n1}$, $[r^{13}, x^{23}] \in F_{n01}$ and $[x^{13}, r^{23}] \in F_{10n}$ and since $n \neq 1$, $\pi_{01n}(\delta_3^{(F)}(x))$ is the same as the image by π_{01n} of $[r^{12}, x^{23}] + [x^{12}, r^{13}] + [r^{12}, x^{13}]$. This is also the image by π_{01n} of

$$\sum_{\sigma \in \mathfrak{S}_n} A_\sigma \mu^{(3)}(x_1^{(2)} \otimes y_1^{(2)} x_{\sigma(1)}^{(1)} \cdots x_{\sigma(n)}^{(1)} \otimes y_1^{(1)} \cdots y_n^{(1)}) \tag{47}$$

$$+ \sum_{\sigma \in \mathfrak{S}_n} A_\sigma \mu^{(3)}(x_{\sigma(1)}^{(2)} \cdots x_{\sigma(n)}^{(2)} \otimes y_1^{(2)} \cdots y_n^{(2)} x_1^{(1)} \otimes y_1^{(1)})$$

$$+ \sum_{\sigma \in \mathfrak{S}_n} A_\sigma [x_1^{(2)}, x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)}] \otimes y_1^{(2)} \otimes y_1^{(3)} \cdots y_n^{(3)},$$

where the argument of the first (resp., second) expression in $\mu^{(3)}$ belongs to $G_{n10}^{(3)}$ (resp., $G_{1n0}^{(3)}$).

The image of (47) by π_{01n} is then equal to

$$\sum_{\sigma \in \mathfrak{S}_n} A_\sigma [x_{\sigma(1)}^{(3)}, \cdots, [x_{\sigma(n)}^{(3)}, x_1^{(2)}]] \otimes y_1^{(2)} \otimes y_1^{(3)} \cdots y_n^{(3)} \tag{48}$$

$$+ \sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma(1)}^{(3)} \cdots [x_1^{(3)}, x_1^{(2)}] \cdots x_{\sigma(n)}^{(3)} \otimes y_1^{(2)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}]$$

$$+ \sum_{\sigma \in \mathfrak{S}_n} A_\sigma [x_1^{(2)}, x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)}] \otimes y_1^{(2)} \otimes y_1^{(3)} \cdots y_n^{(3)}.$$

Lemma D.2. *Let $(X_\sigma)_{\sigma \in \mathfrak{S}_n}$ belong to $\mathbb{K}^{\mathfrak{S}_n}$ and assume that $X = \sum_{\sigma \in \mathfrak{S}_n} X_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is a Lie element of the free algebra with generators $x, x_1, \ldots, x_n$. Then we have*

the identity

$$[X, x] = \sum_{\sigma \in \mathfrak{S}_n} X_\sigma [x_{\sigma(1)}, \cdots, [x_{\sigma(n)}, x]]. \tag{49}$$

Proof of Lemma. Let us denote by ad the adjoint action of the free algebra with $n + 1$ generators on itself by the rule $\mathrm{ad}(a)(b) = [a, b]$. Then we should show the identity $[X, x] = \mathrm{ad}(X)(x)$. X is a linear combination of Lie polynomials of the form $[x_{\tau(1)}, \cdots, [x_{\sigma(n-1)}, x_{\sigma(n)}]]$ (see e.g. [2]), so it suffices to check (49) when X is such a polynomial. Moreover, we may assume after relabeling indices that τ is the inversion $\tau(i) = n + 1 - i$. We have now to prove that if $X_n = [x_n, \cdots, [x_2, x_1]]$, then

$$[X_n, x] = \mathrm{ad}(X_n)(x). \tag{50}$$

Let us prove (50) by induction. The case $n = 1$ is obvious. Assume that (50) holds at step n. Then

$$\mathrm{ad}(X_n)(x) = \mathrm{ad}([x_{n+1}, X_n])(x) = (\mathrm{ad}(x_{n+1})\,\mathrm{ad}(X_n) - \mathrm{ad}(X_n)\,\mathrm{ad}(x_{n+1}))(x)$$
$$= [x_{n+1}, [X_n, x]] - [X_n, [x_{n+1}, x]] = [X_{n+1}, x],$$

where the first equality of the second line uses the twice the induction hypothesis, first with x, then with x replaced by $[x_{n+1}, x]$. $\qquad\square$

Lemma D.2 implies that the first and last terms of (48) cancel out, so that

$$\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma(1)}^{(3)} \cdots x_{\sigma(\sigma^{-1}(1)-1)}^{(3)} \left(x_1^{(3)} x_1^{(2)} - x_1^{(2)} x_1^{(3)} \right) x_{\sigma(\sigma^{-1}(1)+1)}^{(3)} \cdots x_{\sigma(n)}^{(3)}$$
$$\otimes\, y_1^{(2)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}] \tag{51}$$

is zero. Decompose (51) as a sum $\sum_{i=1}^{n+1} A_i$, where A_i contains $x_1^{(2)}$ at the ith position. Then each A_i is zero. Replacing $x_1^{(2)}$ by 1 in each of these equalities, we find the following set of equalities in $(FA_n \otimes FA_n)_{\mathfrak{S}_n}$

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(1)=1} A_\sigma x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}] = 0,$$

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(k)=1} A_\sigma x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}]$$
$$= \sum_{\sigma \in \mathfrak{S}_n | \sigma(k+1)=1} A_\sigma x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}],$$

for $k = 1, \ldots, n - 1$

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(n)=1} A_\sigma x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}] = 0.$$

We have therefore for any $k = 1, \ldots, n$

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(k)=1} A_\sigma x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)} \otimes [[y_1^{(3)}, y_2^{(3)}], \cdots, y_n^{(3)}] = 0.$$

Adding up these equalities, and using the identity $x_{\sigma(1)}^{(3)} \cdots x_{\sigma(n)}^{(3)} \otimes y_1^{(1)} \cdots y_n^{(1)} = x_1^{(3)} \cdots x_n^{(3)} \otimes y_{\sigma^{-1}(1)}^{(1)} \cdots y_{\sigma^{-1}(n)}^{(1)}$ in $(FA_n \otimes FA_n)_{\mathfrak{S}_n}$ we get

$$x_1^{(3)} \cdots x_n^{(3)} \otimes \sum_{\sigma \in \mathfrak{S}_n} A_\sigma [[y_{\sigma^{-1}(1)}^{(3)}, y_{\sigma^{-1}(2)}^{(3)}], \cdots, y_{\sigma^{-1}(n)}^{(3)}] = 0. \tag{52}$$

Proposition D.1. *(see [17]) If $X = \sum_{\sigma \in \mathfrak{S}_n} X_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is a Lie polynomial in the free algebra with generators $x_1, \ldots, x_n$, then*

$$\sum_{\sigma \in \mathfrak{S}_n} X_\sigma [[x_{\sigma(1)}, x_{\sigma(2)}], \cdots, x_{\sigma(n)}] = nX.$$

We have seen that $\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$ is a Lie polynomial, therefore (52) is equal to

$$n \cdot x_1^{(3)} \cdots x_n^{(3)} \otimes \sum_{\sigma \in \mathfrak{S}_n} A_\sigma y_{\sigma^{-1}(1)}^{(3)} \cdots y_{\sigma^{-1}(n)}^{(3)} = nx.$$

Therefore $x = 0$.

D.2. **Computation of H_n^3.**

D.2.1. *Form of the elements of $\operatorname{Im}(\delta_3^{(F)})$.*

Proposition D.2. *Let $(A_\sigma)_{\sigma \in \mathfrak{S}_n} \in \mathbb{K}^{\mathfrak{S}_n}$ be such that $\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is a Lie polynomial. Then for any $k = 1, \ldots, n$, we have*

$$\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma [x_{\sigma(1)}, \cdots [x_{\sigma(n-1)}, x_{\sigma(n)}]].$$

Proof. We may assume that $\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} = [x_1, \ldots, [x_{n-1}, x_n]]$. Let us then prove the result by induction on n. Assume that we proved the result up to order $n - 1$ and let us treat the case of order n. Let us define $(A_\sigma(n))_{\sigma \in \mathfrak{S}_n}$ as the elements of $\mathbb{K}$ such that

$$\sum_{\sigma \in \mathfrak{S}_n} A_\sigma(n) x_{\sigma(1)} \cdots x_{\sigma(n)} = [x_1, \ldots, [x_{n-1}, x_n]].$$

Then if $k = 2, \ldots, n$ and if $\sigma \in \mathfrak{S}_n$ is such that $\sigma(n) = k$ and $A_\sigma(n) \neq 0$, then $\sigma(1) = 1$.

For any $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) = 1$, let us denote by σ' the element of $\mathfrak{S}_{n-1}$ such that $\sigma'(k) = \sigma(k+1) - 1$, for $k = 1, \ldots, n-1$. Then when $\sigma \in \mathfrak{S}_n$ and $\sigma(1) = 1$, we have $A_\sigma(n) = A_{\sigma'}(n-1)$.

Then if $k = 2, \ldots, n$,

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(n) = k} A_\sigma(n) [x_{\sigma(1)}, \ldots, [x_{\sigma(n-1)}, x_{\sigma(n)}]] \tag{53}$$

$$= \sum_{\tau \in \mathfrak{S}_{n-1}} A_\tau(n-1) [x_1, [x_{\sigma'(1)+1}, \ldots, [x_{\sigma'(n-2)+1}, x_{\sigma'(n-1)+1}]]$$

$$= [x_1, [x_2, \ldots, [x_{n-1}, x_n]]],$$

where the second equality follows from the induction hypothesis applied to variables $x_2, \ldots, x_n$. This proves the result at order n, when $k = 2, \ldots, n$. Then

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(n) = 1} A_\sigma(n) [x_{\sigma(1)}, \ldots, [x_{\sigma(n-1)}, x_{\sigma(n)}]]$$

$$= \sum_{\sigma \in \mathfrak{S}_n} A_\sigma(n) [x_{\sigma(1)}, \ldots, [x_{\sigma(n-1)}, x_{\sigma(n)}]] - \sum_{k=2}^{n} \sum_{\sigma \in \mathfrak{S}_n | \sigma(n) = k} A_\sigma(n) [x_{\sigma(1)}, \ldots, [x_{\sigma(n-1)}, x_{\sigma(n)}]]$$

$$= n \sum_{\sigma \in \mathfrak{S}_n} A_\sigma(n) x_{\sigma(1)} \cdots x_{\sigma(n)} - (n-1) \sum_{\sigma \in \mathfrak{S}_n} A_\sigma(n) x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where the second equality follows from Proposition D.1 and from equalities (53). So

$$\sum_{\sigma \in \mathfrak{S}_n | \sigma(n) = 1} A_\sigma(n) [x_{\sigma(1)}, \ldots, [x_{\sigma(n-1)}, x_{\sigma(n)}]] = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma(n) x_{\sigma(1)} \cdots x_{\sigma(n)} = [x_1, \ldots, [x_{n-1}, x_n]],$$

which proves the remaining case of the result at order n. $\qquad\square$

Proposition D.3. *Let $n > 1$ and $y \in F_n$. Set $x = \delta_3^{(F)}(y)$, and let us decompose x as $\sum_{p,q | p+q=n+1} x_{p,q,n+1}^{(aab)} + \sum_{p,q | p+q=n+1} x_{n+1,p,q}^{(abb)}$, with*

$$x_{p,q,n+1}^{(aab)} \in (FL_p \otimes FL_q \otimes FL_{n+1})_{\mathfrak{S}_p \times \mathfrak{S}_q} \quad \text{and} \quad x_{n+1,p,q}^{(abb)} \in (FL_{n+1} \otimes FL_p \otimes FL_q)_{\mathfrak{S}_p \times \mathfrak{S}_q}.$$

Then
1) $x_{n,1,n+1}^{(aab)} = [r^{(23)}, y^{(13)}]$ and $x_{n+1,1,n}^{(abb)} = -[r^{(12)}, y^{(13)}]$.
2) $x_{1,n,n+1}^{(aab)} = 0$ and $x_{n+1,1,n}^{(abb)} = 0$.

Proof. Proposition D.2 allows some simplifications in the computations of the end of Section D.1, which imply the two first equalities. To prove the two last equalities, let us proceed as in Section D.1. For example, if $y = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_1^{(12)} \cdots x_n^{(12)} \otimes y_{\sigma(1)}^{(12)} \cdots y_{\sigma(n)}^{(12)}$, then the nonzero contributions to $x_{1,n,n+1}^{(aab)}$ are those of $[r^{(12)}, y^{(23)}]$ and of $[r^{(13)}, y^{(23)}]$, which are respectively

$$\sum_{k=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} A_\sigma x_1^{(13)} \cdots x_n^{(13)} \otimes x_n^{(23)} y_{\sigma(1)}^{(13)} \cdots [y_{\sigma(k)}^{(13)}, y^{(23)}] \cdots y_{\sigma(n)}^{(13)}$$

and

$$\sum_{k=1}^{n}\sum_{\sigma\in\mathfrak{S}_n} A_\sigma x_1^{(13)}\cdots x_n^{(13)}\otimes x^{(23)}[y^{(23)},y_{\sigma(1)}^{(13)}\cdots y_{\sigma(n)}^{(13)}]$$

and cancel out (here we do not use the fact that y belongs to $(FL_n\otimes FL_n)_{\mathfrak{S}_n}$). $\qquad\square$

D.2.2. *Computation of H_n^3.* The result is obvious in the case $n=2$.

Let us assume that $n>2$ and let x belong to $F_n^{Lie,(3)}$ be such that $\delta_4^{(F)}(x)=0$. We want to show the existence of $y\in F_{n-1}^{Lie,(2)}$ such that $x=\delta_3^{(F)}(y)$.

Proposition D.4. *We have $x_{1,n-1,n}^{(aab)}=0$ and $x_{n,n-1,1}^{(abb)}=0$.*

Proof. When $z\in F_n^{Lie,(4)}$, $x,y\in\{a,b\}$ and p,q,r,s are integers, we denote by $z_{pqrs}^{(axyb)}$ the projection of z on $F_{pqrs}^{(axyb)}$ parallel to the direct sum of all other $F_{p'q'r's'}^{(abx'y'b)}$. Then $(\delta_4^{(F)}(x))_{1,1,n-1,n+1}^{(aaab)}$ is equal to

$$\big([r^{(12)}+r^{(13)}+r^{(14)},(x_{1,n-1,n}^{(aab)})^{(234)}]-[r^{(23)}+r^{(24)},(x_{1,n-1,n}^{(aab)})^{(134)}]\big)_{1,1,n-1,n+1}^{(aaab)},$$

because all other summands of $\delta_4^{(F)}(x)$ project to zero on $F_{1,1,n-1,n+1}^{(aaab)}$, either obviously, or in the case of $[r^{(34)},(x_{112}^{(aab)})^{(123)}+(x_{112}^{(aab)})^{(124)}]$, because we assumed that $n>2$.

Lemma D.3.
$$\big([r^{(12)}+r^{(13)}+r^{(14)},(x_{1,n-1,n}^{(aab)})^{(234)}]\big)_{1,1,n-1,n+1}^{(aaab)}=0$$

Proof of Lemma. Let $(Z_{k,\sigma})_{(k,\sigma)\in\{0,\dots,n-1\}\times\mathfrak{S}_{n-1}}$ be the elements of $\mathbb{K}$ such that

$$x_{1,n-1,n}^{(aab)}=\sum_{k=0}^{n-1}\sum_{\sigma\in\mathfrak{S}_{n-1}} Z_{k,\sigma}x_1^{(3)}\otimes x_1^{(1)}\cdots x_{n-1}^{(1)}\otimes y_{\sigma(1)}^{(1)}\cdots y_{\sigma(k)}^{(1)}y_1^{(3)}y_{\sigma(k+1)}^{(1)}\cdots y_{\sigma(n-1)}^{(1)}.$$

Then

$$[r^{(12)},(x_{1,n-1,n}^{(aab)})^{(234)}]$$

$$=\sum_{k=0}^{n-1}\sum_{\sigma\in\mathfrak{S}_{n-1}} Z_{k,\sigma}x_1^{(14)}\otimes x_1^{(24)}\otimes x_1^{(34)}\cdots x_{n-1}^{(34)}$$

$$\otimes y_{\sigma(1)}^{(34)}\cdots y_{\sigma(k)}^{(34)}[y_1^{(24)},y_1^{(14)}]y_{\sigma(k+1)}^{(34)}\cdots y_{\sigma(n-1)}^{(34)}$$

and

$$[r^{(13)},(x_{1,n-1,n}^{(aab)})^{(234)}]$$

$$=\sum_{k=0}^{n-1}\sum_{\sigma\in\mathfrak{S}_{n-1}} Z_{k,\sigma}\sum_{s=1}^{n-1} x_1^{(14)}\otimes x_1^{(24)}\otimes x_1^{(34)}\cdots x_{n-1}^{(34)}$$

$$\otimes y_{\sigma(1)}^{(34)}\cdots[y_{\sigma(s)}^{(34)},y_1^{(14)}]\cdots y_{\sigma(k)}^{(34)}y_1^{(24)}y_{\sigma(k+1)}^{(34)}\cdots y_{\sigma(n-1)}^{(34)},$$

so that the sum of these terms is equal to $-[r^{(14)}, (x_{1,n-1,n}^{(aab)})^{(234)}]$. □

End of proof of Proposition. In the same way, one proves

$$\left([r^{(23)} + r^{(24)}, (x_{1,n-1,n}^{(aab)})^{(234)}]\right)_{1,1,n-1,n+1}^{(aaab)}$$

$$= \sum_{k=0}^{n-1} \sum_{\sigma \in \mathfrak{S}_{n-1}} Z_{k,\sigma} x_1^{(14)} \otimes x_1^{(24)} \otimes x_1^{(34)} \cdots x_{n-1}^{(34)}$$

$$\otimes y_{\sigma(1)}^{(34)} \cdots y_{\sigma(k)}^{(34)} [y_1^{(14)}, y_1^{(24)}] y_{\sigma(k+1)}^{(34)} \cdots y_{\sigma(n-1)}^{(34)}.$$

Therefore $\sum_{k=0}^{n-1} \sum_{\sigma \in \mathfrak{S}_{n-1}} Z_{k,\sigma} y_{\sigma(1)}^{(34)} \cdots y_{\sigma(k)}^{(34)} [y_1^{(14)}, y_1^{(24)}] y_{\sigma(k+1)}^{(34)} \cdots y_{\sigma(n-1)}^{(34)} = 0$. For any k, the sum of all the terms in this sum in which $y_1^{(14)}$ and $y_1^{(24)}$ appear in the kth and $(k+1)$st position is also zero, so for any k we have $\sum_{\sigma \in \mathfrak{S}_{n-1}} Z_{k,\sigma} y_{\sigma(1)}^{(34)} \cdots y_{\sigma(n-1)}^{(34)} = 0$. So the $Z_{k,\sigma}$ are all zero and $x_{1,n-1,n}^{(aab)} = 0$.

The proof of $x_{n,n-1,1}^{(abb)} = 0$ is similar. □

Recall that $F^{Lie,(n)}$ is the direct sum $\bigoplus_{x_1,\ldots,x_{n-2} \in \{a,b\}} F^{(ax_1 \ldots x_{n-2} b)}$. If $z \in F^{Lie,(n)}$, let us denote by $x^{(ax_1 \ldots x_{n-2} b)}$ the projection of z on $F^{(ax_1 \ldots x_{n-2} b)}$ parallel to the direct sum of all other $F^{(ax_1' \ldots x_{n-2}' b)}$.

Proposition D.5. *If z belongs to $\bigoplus_{x \in \{a,b\}} F_n^{(axb)}$, then*

$$(\delta_4(z))^{(aabb)} = -[r^{(13)}, (z^{(aab)})^{(124)} + (z^{(aab)})^{(214)}] - [r^{(24)}, (z^{(abb)})^{(134)} + (z^{(abb)})^{(314)}].$$

Proof. If n is an integer ≥ 2, $x_1, \ldots, x_{n-3}$ belong to $\{a, b\}$, and if $1 \leq i < j \leq n$, then

$$[r^{(ij)}, (F^{(ax_1 \cdots x_{n-3} b)})^{1,\ldots,i-1,i+1,\ldots,n}] \subset \bigoplus_{x \in \{a,b\}} F^{(ax_1 \cdots x_{i-2} a x_{i-1} \cdots x_{j-3} x x_{j-2} \cdots x_{n-3} b)}$$

and

$$[r^{(ij)}, (F^{(ax_1 \cdots x_{n-3} b)})^{1,\ldots,j-1,j+1,\ldots,n}] \subset \bigoplus_{x \in \{a,b\}} F^{(ax_1 \cdots x_{i-2} a x_{i-1} \cdots x_{j-3} x x_{j-2} \cdots x_{n-3} b)}.$$

This implies that $(\delta_4^{(F)}(z))^{(aabb)}$ is equal to

$$\left([r^{(12)} + r^{(13)} + r^{(14)}, (z^{(abb)})^{(234)}] + [r^{(13)}, (z^{(aab)})^{(234)}] - [r^{(23)}, (z^{(aab)})^{(134)}]\right.$$

$$- [r^{(23)} + r^{(24)}, (z^{(abb)})^{(134)}] - [r^{(13)} + r^{(23)}, (z^{(aab)})^{(124)}] - [r^{(23)}, (z^{(abb)})^{124}]$$

$$\left. + [r^{(14)} + r^{(24)} + r^{(34)}, (z^{(abb)})^{(123)}] + [r^{(14)}, (z^{(abb)})^{(123)}]\right)^{(aabb)}. \tag{54}$$

The reasoning of Proposition D.3, 2) implies that

$$\left([r^{(12)} + r^{(13)} + r^{(14)}, (z^{(abb)})^{(234)}]\right)^{(aabb)} \quad \text{and} \quad \left([r^{(14)} + r^{(24)} + r^{(34)}, (z^{(abb)})^{(123)}]\right)^{(aabb)}$$

are zero (here we no not use the fact that the components of $z^{(aab)}$ and $z^{(abb)}$ are Lie polynomials) and the reasoning of Proposition D.3, 1) relying on Proposition D.2 and the fact that the components of $z^{(aab)}$ and $z^{(abb)}$, shows that

$$([r^{(23)}, (z^{(aab)})^{(134)}])^{(aabb)} = -[r^{(23)}, (z^{(aab)})^{(124)}]$$

and

$$([r^{(23)}, (z^{(abb)})^{(124)}])^{(aabb)} = -[r^{(23)}, (z^{(abb)})^{(134)}].$$

Permuting the two first tensor factors of these relations, we find

$$([r^{(13)}, (z^{(aab)})^{(134)}])^{(aabb)} = -[r^{(13)}, (z^{(aab)})^{(214)}]$$

and

$$([r^{(23)}, (z^{(abb)})^{(123)}])^{(aabb)} = -[r^{(23)}, (z^{(abb)})^{(143)}].$$

Substituting these expressions in (54) gives the result. $\square$

Remark 14. One proves in the same way that $(\delta_4^{(F)}(z))^{(abab)}$ is identically zero.

$\square$

Corollary D.1. *Recall that x belongs to $F_n^{Lie,(3)}$ and is such that $\delta_4^{(F)}(x) = 0$. Then there exists $y \in F_{n-1}$ such that*

$$x_{n-1,1,n}^{(aab)} = [r^{(23)}, y^{(13)}] \quad \text{and} \quad x_{n,1,n-1}^{(abb)} = -[r^{(12)}, y^{(13)}].$$

Moreover, for any integers p, q such that $p + q = n$ and $p, q > 1$, we have

$$x_{p,q,n}^{(aab)} + (x_{q,p,n}^{(aab)})^{(213)} = 0, \quad x_{n,p,q}^{(abb)} + (x_{n,q,p}^{(abb)})^{(132)} = 0.$$

Proof. It follows from Proposition D.5 that

$$[r^{(13)}, (x^{(aab)})^{(124)} + (x^{(aab)})^{(214)}] + [r^{(24)}, (x^{(abb)})^{(134)} + (x^{(abb)})^{(143)}] = 0. \tag{55}$$

Let us project this equation on $F_{n,1,1,n}^{(aabb)}$ parallel to the sum of all other $F_{pqrs}^{(aabb)}$. Since $x_{1,n-1,n}^{(aab)} = 0$ and $x_{n,n-1,1}^{(abb)} = 0$, we get

$$[r^{(13)}, (x_{n-1,1,n}^{(aab)})^{(124)}] + [r^{(24)}, (x_{n,1,n-1}^{(abb)})^{(134)}] = 0.$$

Let us set

$$x_{n-1,1,n}^{(aab)} = \sum_{k=0}^{n-1} \sum_{\sigma \in \mathfrak{S}_{n-1}} A_{k,\sigma} x_1^{(3)} \cdots x_{n-1}^{(3)} \otimes x_1^{(1)} \otimes y_{\sigma(1)}^{(3)} \cdots y_{\sigma(k)}^{(3)} y_1^{(1)} y_{\sigma(k+1)}^{(3)} \cdots y_{\sigma(n-1)}^{(3)}$$

and

$$x_{n,1,n-1}^{(abb)} = \sum_{k=0}^{n-1} \sum_{\sigma \in \mathfrak{S}_{n-1}} B_{k,\sigma} x_{\sigma(1)}^{(3)} \cdots x_{\sigma(k)}^{(3)} x_1^{(2)} x_{\sigma(k+1)}^{(3)} \cdots x_{\sigma(n-1)}^{(3)} \otimes y_1^{(2)} \otimes y_1^{(3)} \cdots y_{n-1}^{(3)}.$$

Then we find

$$\sum_{k=0}^{n-1}\sum_{\sigma\in\mathfrak{S}_{n-1}}A_{k,\sigma}[x_1^{(13)},x_1^{(14)}\cdots x_{n-1}^{(14)}]\otimes x_1^{(24)}\otimes y_1^{(13)}\otimes y_{\sigma(1)}^{(14)}\cdots y_{\sigma(k)}^{(14)}y_1^{(24)}y_{\sigma(k+1)}^{(14)}\cdots y_{\sigma(n-1)}^{(14)}$$

$$+\sum_{k=0}^{n-1}\sum_{\sigma\in\mathfrak{S}_{n-1}}x_{\sigma(1)}^{(14)}\cdots x_{\sigma(k)}^{(14)}x_1^{(13)}x_{\sigma(k+1)}^{(14)}\cdots x_{\sigma(n-1)}^{(14)}\otimes x_1^{(24)}\otimes y_1^{(13)}\otimes[y_1^{(24)},y_1^{(14)}\cdots y_{n-1}^{(14)}]=0.$$

Identifying terms, we find that $A_{k,\sigma}=B_{k,\sigma}=0$ whenever $k\notin\{0,n-1\}$. Moreover, for any $\sigma\in\mathfrak{S}_{n-1}$, we get

$$A_{0,\sigma}=-B_{0,\sigma^{-1}},\ \ A_{n-1,\sigma}=B_{0,\sigma^{-1}},\ \ A_{0,\sigma}=B_{n-1,\sigma^{-1}},\ \ -A_{n-1,\sigma}=-B_{n-1,\sigma^{-1}}.$$

Let us set $C_\sigma=A_{0,\sigma}$, then $A_{0,\sigma}=-A_{n-1,\sigma}=C_\sigma$ and $B_{0,\sigma^{-1}}=-B_{n-1,\sigma^{-1}}=-C_\sigma$, so if we set

$$y=\sum_{\sigma\in\mathfrak{S}_{n-1}}x_1^{(12)}\cdots x_{n-1}^{(12)}\otimes y_{\sigma(1)}^{(12)}\cdots y_{\sigma(n-1)}^{(12)},$$

$y\in F^{(2)}$ and we get

$$x_{n-1,1,n}^{(aab)}=[r^{(23)},y^{(13)}]\quad\text{and}\quad x_{n,1,n-1}^{(abb)}=-[r^{(12)},y^{(13)}].$$

Let us show that $y\in F_{n-1}=(FL_{n-1}\otimes FL_{n-1})_{\mathfrak{S}_{n-1}}$. Since $x_{n-1,1,n}^{(aab)}$ and $x_{n,1,n-1}^{(abb)}$ belong to $F_{n-1,n,1}^{(aab)}$ and $F_{1,n-1,n}^{(abb)}$, the commutators $[y_1^{(23)},\sum_{\sigma\in\mathfrak{S}_{n-1}}C_\sigma y_{\sigma(1)}^{(13)}\cdots y_{\sigma(n-1)}^{(13)}]$ and $[x_1^{(12)},\sum_{\sigma\in\mathfrak{S}_{n-1}}C_{\sigma^{-1}}x_{\sigma(1)}^{(13)}\cdots x_{\sigma(n-1)}^{(13)}]$ are Lie polynomials. The first statement implies that $\sum_{\sigma\in\mathfrak{S}_{n-1}}C_\sigma y_{\sigma(1)}^{(13)}\cdots y_{\sigma(n-1)}^{(13)}$ is a Lie polynomial, and the second statement implies that $\sum_{\sigma\in\mathfrak{S}_{n-1}}C_{\sigma^{-1}}x_{\sigma(1)}^{(13)}\cdots x_{\sigma(n-1)}^{(13)}$ is also a Lie polynomial; by virtue of Lemma D.1, this implies that y belongs to F_{n-1}.

To prove the second part of the Proposition, let us now project equation (55) on $F_{p+1,q,1,n}^{(aabb)}$. Since $q\neq 1$, the contribution of the second term of (55) is zero, so that $[r^{(13)},(x^{(aab)})_{p,q,n}^{(124)}+(x^{(aab)})_{p,q,n}^{(214)}]=0$, therefore

$$x_{p,q,n}^{(aab)}+(x^{(aab)})_{p,q,n}^{(213)}=0.$$

$\square$

Let us set now $x'=x-\delta_3^{(F)}(x)$.

Proposition D.6. *We have $x'=0$.*

Proof. Let us summarize the properties of x'. We have $\delta_F^{(4)}(x')=0$, $(x')_{n-1,1,n}^{(aab)}=(x')_{1,n-1,n}^{(aab)}=0$, $(x')_{n,1,n-1}^{(abb)}=(x')_{n,n-1,1}^{(abb)}=0$ and for any pair of integers p,q such that $p+q=n$, $(x')_{p,q,n}^{(aab)}+((x')^{(aab)})_{p,q,n}^{(213)}=0$ and $(x')_{n,p,q}^{(abb)}+((x')^{(abb)})_{n,p,q}^{(132)}=0$.

The first property follows from $\delta_4^{(F)}\circ\delta_3^{(F)}=0$, the second property follows from Proposition D.3, 1), and the third property is a consequence of the first property and Proposition D.1.

Let k be an integer such that $1 < k < n$, and let us project the equality $\delta_4^{(F)}(x') = 0$ to $F_{k,n-k,1,n+1}^{(aaab)}$ parallel to all other $F_{pqrs}^{(axyb)}$. We have

$$[r^{(12)}, (F^{(abb)})^{(234)}] \subset F^{(aabb)} \oplus F^{(abbb)},$$

and

$$[r^{(12)}, (F_{pqn}^{(aab)})^{(234)}] \subset \bigoplus_{k=1}^{p} F_{k,p+1-k,q,n+1}^{(aaab)} \oplus F_{p+1,1,q,n}^{(abab)},$$

so that for any $z \in F_n^{Lie,(4)}$,

$$(\delta_4^{(F)}(z))_{k,n-k,1,n+1}^{(aaab)} = \left([r^{(12)}, (z_{n-1,1,n}^{(aab)})^{(234)}] + [r^{(13)}, (z_{n-k,k,n}^{(aab)})^{(234)}] - [r^{(23)}, (z_{k,n-k,n}^{(aab)})^{(134)}] \right.$$
$$\left. + [r^{(34)}, (z_{k,n-k,n}^{(aab)})^{(124)}] + [r^{(34)}, (z_{k,n-k,n}^{(aab)})^{(123)}]\right)_{k,n-k,1,n+1}^{(aaab)}.$$

Lemma D.4. *For any $w \in F_{k,n-k,n}^{(aab)}$, we have*

$$\left([r^{(13)}, w^{(324)}] + [r^{(23)}, w^{(134)}]\right)_{k,n-k,1,n+1}^{(aaab)} = [r^{(34)}, w^{(124)}].$$

Proof. There exists a unique family $(A_{\sigma,\tau,\tau'})$ in $\mathbb{K}^{\mathfrak{S}_{k,n-k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k}}$, such that

$$w = \sum_{(\sigma,\tau,\tau') \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k}} A_{\sigma,\tau,\tau'} x_{\tau(1)}^{(13)} \cdots x_{\tau(k)}^{(13)} \otimes x_{\tau'(1)}^{(23)} \cdots x_{\tau'(n-k)}^{(23)} \otimes y_{\sigma(1)} \cdots y_{\sigma(n)},$$

where $(y_1, \ldots, y_n) = (y_1^{(13)}, \ldots, y_k^{(13)}, y_1^{(23)}, \ldots, y_{n-k}^{(23)})$. Then for each $(\sigma, \tau') \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_{n-k}$, $\sum_{\tau \in \mathfrak{S}_k} A_{\sigma,\tau,\tau'} x_{\tau(1)}^{(13)} \cdots x_{\tau(k)}^{(13)}$ is a Lie polynomial, and for each $(\sigma, \tau) \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_k$, $\sum_{\tau' \in \mathfrak{S}_k} A_{\sigma,\tau,\tau'} x_{\tau'(1)}^{(23)} \cdots x_{\tau'(n-k)}^{(23)}$ is also a Lie polynomial. Then

$$([r^{(23)}, w^{(134)}])_{k,n-k,1,n+1}^{(aaab)}$$
$$= \sum_{(\sigma,\tau,\tau') \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k}} A_{\sigma,\tau,\tau'} \mu^{(4)}\left(x_{\tau(1)}^{(14)} \cdots x_{\tau(k)}^{(14)} \otimes x_1^{(23)} \otimes [y_1^{(23)}, x_{\tau'(1)}^{(34)} \cdots x_{\tau'(n-k)}^{(34)}]\right)$$
$$\otimes y_{\sigma(1)}' \cdots y_{\sigma(n)}')_{k,n-k,1,n+1}^{(aaab)}$$
$$= \sum_{(\sigma,\tau,\tau') \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k}} A_{\sigma,\tau,\tau'} x_{\tau(1)}^{(14)} \cdots x_{\tau(k)}^{(14)} \otimes [x_{\tau'(1)}^{(24)}, \cdots [x_{\tau'(n-k-1)}^{(24)}, x_1^{(24)}]] \otimes y_1^{(23)}$$
$$\otimes y_{\sigma(1)}'' \cdots [y_{k+\tau'(n-k)}'', y_1^{(24)}] \cdots y_{\sigma(n)}'',$$

where $(y_1', \ldots, y_n') = (y_1^{(14)}, \ldots, y_k^{(14)}, y_1^{(34)}, \ldots, y_{n-k}^{(34)})$ and

$$(y_1'', \ldots, y_n'') = (y_1^{(13)}, \ldots, y_k^{(13)}, y_1^{(23)}, \ldots, y_{n-k}^{(23)}).$$

Now Lemma D.2 and the fact that $\sum_{\tau' \in \mathfrak{S}_k} A_{\sigma,\tau,\tau'} x^{(23)}_{\tau'(1)} \cdots x^{(23)}_{\tau'(n-k)}$ is a Lie polynomial implies that

$$\left([r^{(23)}, w^{(134)}]\right)^{(aaab)}_{k,n-k,1,n+1} \tag{56}$$

$$= \sum_{(\sigma,\tau,\tau') \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k}} A_{\sigma,\tau,\tau'} \sum_{i|\sigma(i) \in \{k+1,\dots,n\}} x^{(14)}_{\tau(1)} \cdots x^{(14)}_{\tau(k)} \otimes x^{(24)}_{\tau'(1)} \cdots x^{(24)}_{\tau'(n-k)} \otimes y^{(23)}_1$$

$$\otimes\, y''_{\sigma(1)} \cdots [y^{(24)}_1, y''_{\sigma(i)}] \cdots y''_{\sigma(n)}.$$

Applying $x \mapsto x^{(2134)}$ to the equality (56), where k and w are replaced by $n-k$ and $w^{(213)}$ yields

$$\left([r^{(23)}, w^{(134)}]\right)^{(aaab)}_{k,n-k,1,n+1} \tag{57}$$

$$= \sum_{(\sigma,\tau,\tau') \in \mathfrak{S}_{k,n-k} \times \mathfrak{S}_k \times \mathfrak{S}_{n-k}} A_{\sigma,\tau,\tau'} \sum_{i|\sigma(i) \in \{1,\dots,k\}} x^{(14)}_{\tau(1)} \cdots x^{(14)}_{\tau(k)} \otimes x^{(24)}_{\tau'(1)} \cdots x^{(24)}_{\tau'(n-k)} \otimes y^{(23)}_1$$

$$\otimes\, y''_{\sigma(1)} \cdots [y^{(24)}_1, y''_{\sigma(i)}] \cdots y''_{\sigma(n)}.$$

The result now follows from the addition of (56) and (57). □

Lemma D.5. *For any $w \in F^{(aab)}_{k,n-k,n}$, we have*

$$\left([r^{(34)}, w^{(123)} + w^{(124)}]\right)^{(aaab)}_{k,n-k,1,n+1} = 0.$$

Proof. There exists a unique family $(A_\sigma)_{\sigma \in \mathfrak{S}_n} \in \mathbb{K}^{\mathfrak{S}_n}$, such that

$$w = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma x^{(13)}_1 \cdots x^{(13)}_k \otimes x^{(13)}_{k+1} \cdots x^{(13)}_n \otimes y_{\sigma(1)} \cdots y_{\sigma(n)},$$

where $(y_1, \dots, y_n) = (y^{(13)}_1, \dots, y^{(13)}_k, y^{(23)}_1, \dots, y^{(23)}_n)$. Then $\sum_{\sigma \in \mathfrak{S}_n} A_\sigma y_{\sigma(1)} \cdots y_{\sigma(n)}$ is a Lie polynomial. We have

$$[r^{(34)}, w^{(123)}] = \mu^{(4)} \Big(\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x^{(13)}_1 \cdots x^{(13)}_k \otimes x^{(23)}_{k+1} \cdots x^{(23)}_n \otimes [x^{(34)}_1, y_{\sigma(1)} \cdots y_{\sigma(n)}] \otimes y^{(34)}_1 \Big)$$

$$= \mu^{(4)} \Big(\sum_{\sigma \in \mathfrak{S}_n} A_\sigma x^{(13)}_1 \cdots x^{(13)}_k \otimes x^{(23)}_{k+1} \cdots x^{(23)}_n \otimes x^{(34)}_1 \otimes [y'_{\sigma(1)}, \cdots, [y'_{\sigma(n)}, y^{(34)}_1]] \Big)$$

$$= -[r^{(34)}, w^{(124)}]$$

where $(y'_1, \dots, y'_n) = (y^{(14)}_1, \dots, y^{(14)}_k, y^{(24)}_1, \dots, y^{(24)}_n)$. The last equality follows from Lemma D.2 and the fact that $\sum_{\sigma \in \mathfrak{S}_n} A_\sigma y_{\sigma(1)} \cdots y_{\sigma(n)}$ is a Lie polynomial. □

End of proof of Proposition D.6. By virtue of the two Lemmas above, and since $(x')^{(aab)}_{n-1,1,n} = 0$, and $(x')^{(aab)}_{k,n-k,n} + ((x')^{(aab)}_{n-k,k,n})^{(213)} = 0$, $(\delta^{(F)}_4(x'))^{(aaab)}_{k,n-k,1,n+1} = 0$ yields

$$[r^{(34)}, (x')^{(aab)}_{k,n-k,n}] = 0,$$

which implies that $(x')^{(aab)}_{k,n-k,n} = 0$, so $(x')^{(aab)} = 0$. One shows in the same way that $(x')^{(abb)} = 0$, so $x' = 0$. □

Remark 15. The projection of $\delta_4^{(F)}(x') = 0$ to $F^{(aaab)}_{k,1,n-k,n+1}$ also yields the result. On the other hand, the projection of $\delta_4^{(F)}(x)$ to $F^{(aaab)}_{1,k,n-k,n+1}$ is identically zero. □

End of the computation of H_n^3. We have shown that for any $n > 2$ and $x \in F_n^{Lie,(3)}$ such that $\delta_4^{(F)}(x) = 0$, there exists $y \in F_{n-1}$ such that $x = \delta_3^{(F)}(y)$. Therefore $H_n^3 = 0$. □

Appendix E. Universal shuffle algebras (proof of Theorem 3.2)

In this Section, we define universal shuffle algebras $\mathrm{Sh}_k^{(F)}$. These algebras have universal properties with respect to the tensor powers $\mathrm{Sh}(\mathfrak{a})^{\otimes k}$, where $\mathfrak{a}$ is a Lie algebra endowed with a solution $r_\mathfrak{a} \in \mathfrak{a} \otimes \mathfrak{a}$ of CYBE.

E.1. **Definition of** $\mathrm{Sh}_k^{(F)}$. Let k be an integer ≥ 0. We set $\mathrm{Sh}_k^{(F)} = \mathbb{K}$ for $k = 0$ and $k = 1$. When $k \geq 2$, we put the following definitions. If $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we define $X(\underline{\alpha})$ as the set of all maps $x : I_{\underline{\alpha}} \to \{a, b\}$ (recall that $I_{\underline{\alpha}} = \{(i, \beta) | i \in \{1, \ldots, n\}, \beta \in \{1, \ldots, \alpha_i\}\}$). If $x \in X(\underline{\alpha})$, let us denote by $P(x, \underline{\alpha})$ the set of all maps $\underline{p} : x^{-1}(a) \times x^{-1}(b) \to \mathbb{N}$, such that for any $(i, \beta) \in x^{-1}(a)$ and $(i', \beta') \in x^{-1}(b)$, $\underline{p}((i, \beta), (i', \beta')) = 0$ whenever $i = i'$ or $(i, \beta) > (i', \beta')$ in the lexicographical order.

We then set

$$\mathrm{Sh}_k^{(F)} = \bigoplus_{\underline{\alpha} \in \mathbb{N}^k} \Big(\bigoplus_{x \in X(\underline{\alpha})} \Big(\bigoplus_{\underline{p} \in P(x, \underline{\alpha})} F^{(x(1,1)\ldots x(1,\alpha_1) x(2,1)\ldots x(k,\alpha_k))}_{\left(\underline{p}((i,\beta),(i',\beta'))\right)_{((i,\beta),(i',\beta')) \in x^{-1}(a) \times x^{-1}(b)}} \Big) \Big), \tag{58}$$

where the space $F^{(x_1 \ldots x_n)}_{(p(k,l))_{(k,l) \in K \times L}}$ is defined by (18). The summand corresponding by $\underline{\alpha} = 0$ is $\mathbb{K}$. (For each i, one should think of the tensor product of all the tensor factors indexed by (i, α) as of the analogue of the ith factor of $\mathrm{Sh}(\mathfrak{a})^{\otimes k}$.)

We denote by $\mathrm{Sh}_k^{(F)}(\underline{\alpha})$ (resp., $\mathrm{Sh}_k^{(F)}(\underline{\alpha}, x)$, $\mathrm{Sh}_k^{(F)}(\underline{\alpha}, x, \underline{p})$) the graded component of $\mathrm{Sh}_k^{(F)}$ corresponding to $\underline{\alpha}$ (resp., to $(\underline{\alpha}, x)$, $(\underline{\alpha}, x, \underline{p})$).

E.2. **Operations of** $\mathrm{Sh}_k^{(F)}$.

E.2.1. *The multiplication* $m_{\mathrm{Sh}_k^{(F)}}$. Let us first define a bilinear map

$$\ell : (\otimes_{i=1}^l F L_{\alpha_i + \alpha_i'}) \times (\mathrm{Sh}_k^{(F)}(\underline{\alpha}) \otimes \mathrm{Sh}_k^{(F)}(\underline{\alpha}')) \to \mathrm{Sh}_k^{(F)}(\underline{1}),$$

where $\underline{1}$ is the element of $\mathbb{N}^k$ with all components equal to 1.

Assume that u and v are decomposed as $u = \otimes_{i=1}^k (\otimes_{\beta=1}^{\alpha_i} u_{i,\beta})$ and $v = \otimes_{i=1}^k (\otimes_{\beta=1}^{\alpha_i'} v_{i,\beta})$, then we view $\otimes_{i=1}^k L_i(u_{i,1}, \ldots, u_{i,\beta_i}, v_{i,1}, \ldots, v_{i,\beta_i'})$ as an element of $G^{(k)}$. Then it follows from Corollary C.4 that $\mu^{(k)}(\otimes_{i=1}^k L_i(u_{i,1}, \ldots, u_{i,\beta_i}, v_{i,1}, \ldots, v_{i,\beta_i'}))$ belongs to $\oplus_{x \in \mathrm{Map}(\{1,\ldots,k\},\{a,b\})} F^{(x(1)\ldots x(k))}_{\underline{p}}$. The latter space is exactly $\mathrm{Sh}_k^{(F)}(\underline{1})$, and we set

$$\ell(\otimes_{i=1}^k L_i, u \otimes v) = \mu^{(k)}(\otimes_{i=1}^k L_i(u_{i,1}, \ldots, u_{i,\beta_i}, v_{i,1}, \ldots, v_{i,\beta_i'})).$$

Let us fix $\underline{\gamma} \in \mathbb{N}_{>0}^k$ and for each i, let us fix γ_i-partitions of α_i and α_i', namely $\alpha_i = \alpha_{i1} + \cdots + \alpha_{i\gamma_i}$ and $\alpha_i' = \alpha_{i1}' + \cdots + \alpha_{i\gamma_i}'$. Then we define a bilinear map

$$\ell_{\underline{\alpha}, \underline{\alpha}', \underline{\gamma}, (\alpha_{ij}), (\alpha_{ij}')} : \big(\otimes_{i=1}^l (\otimes_{\beta=1}^{\gamma_i} F L_{\alpha_{i\beta} + \alpha_{i\beta}'}) \big) \times (\mathrm{Sh}_k^{(F)}(\underline{\alpha}) \otimes \mathrm{Sh}_k^{(F)}(\underline{\alpha}')) \to \mathrm{Sh}_k^{(F)}(\underline{\gamma})$$

as follows. Let us define $\Delta_{\underline{\gamma}, (\alpha_{ij})}$ as the linear map

$$\Delta_{\underline{\gamma}, (\alpha_{ij})} : \mathrm{Sh}_k^{(F)}(\underline{\alpha}) \to \mathrm{Sh}_{\sum_i \gamma_i}^{(F)}(\alpha_1 1, \ldots, \alpha_{1,\gamma_1}, \alpha_{21}, \ldots, \alpha_{k\gamma_k})$$

defined as the canonical injection of $\mathrm{Sh}_k^{(F)}(\underline{\alpha}, x, \underline{p})$ into $\mathrm{Sh}_{\sum_i \gamma_i}^{(F)}((\alpha_{ij})_{(i,j)|1 \leq i \leq n, 1 \leq j \leq \gamma_i}, x', \underline{p}')$, where x' is the composition of x with the lexicographical bijection $I_{\alpha_{11}, \ldots, \alpha_{k\gamma_k}} \to I_{\underline{\alpha}}$, and $\underline{p}'$ is the composition of $\underline{p}$ with the square of this bijection.

In the same way, $\Delta_{1,\underline{1}}$ is an injection of $\mathrm{Sh}_k^{(F)}(\underline{\gamma})$ in $\mathrm{Sh}_{\sum_i \gamma_i}^{(F)}(\underline{1})$, associated with the partitions $(1, \ldots, 1)$ (γ_i times 1) of each γ_i.

Then if $u \in \mathrm{Sh}_k^{(F)}(\underline{\alpha})$ and $v \in \mathrm{Sh}_k^{(F)}(\underline{\alpha}')$, and if for each $(i, \beta) \in I_{\underline{\gamma}}$, $L_{i,\beta} \in FL_{\alpha_{i\beta} + \alpha'_{i\beta}}$, then

$$\ell\big(\otimes_{i=1}^{k} (\otimes_{\beta=1}^{\gamma_i} L_{i\beta}), \Delta_{\underline{\gamma}, (\alpha_{ij})}(u) \otimes \Delta_{\underline{\gamma}, (\alpha'_{ij})}(v)\big)$$

is in the image of $\Delta_{1,\underline{1}}$, and we denote by

$$\ell_{\underline{\alpha}, \underline{\alpha}', \underline{\gamma}, (\alpha_{ij}), (\alpha'_{ij})}\big(\otimes_{i=1}^{k} (\otimes_{\beta=1}^{\gamma_i} L_{i\beta}), u \otimes v\big)$$

as the preimage of this element by $\Delta_{1,\underline{1}}$.

Then there exists a unique linear map $m_{\mathrm{Sh}_k^{(F)}} : \mathrm{Sh}_k^{(F)} \otimes \mathrm{Sh}_k^{(F)} \to \mathrm{Sh}_k^{(F)}$, such that if $\underline{\alpha} \in \mathbb{N}^k$ and $\underline{\alpha}' \in \mathbb{N}^{k'}$, and if $u \in \mathrm{Sh}_k^{(F)}(\underline{\alpha})$ and $u' \in \mathrm{Sh}_k^{(F)}(\underline{\alpha}')$,

$$m_{\mathrm{Sh}_k^{(F)}}(u \otimes v) = \sum_{\underline{\gamma} \in \mathbb{N}^k} \sum_{(\alpha_{ij}) \in P(\underline{\alpha}, \underline{\gamma}), (\alpha'_{ij}) \in P(\underline{\alpha}', \underline{\gamma})}$$

$$\ell_{\underline{\alpha}, \underline{\alpha}', \underline{\gamma}, (\alpha_{ij}), (\alpha'_{ij})}\big(\otimes_{i=1}^{k} (\otimes_{\beta=1}^{\gamma_i} B_{\alpha_{i\beta}, \alpha'_{i\beta}}), u \otimes v\big),$$

where $P(\underline{\alpha}, \underline{\gamma})$ is the set of collections $\big((\alpha_{1j})_{j=1, \ldots, \gamma_1}, \ldots, (\alpha_{kj})_{j=1, \ldots, \gamma_k}\big)$, where for each i, $(\alpha_{ij})_{j=1, \ldots, \gamma_i}$ is a γ_i-partition of α_i.

Proposition E.1. $m_{\mathrm{Sh}_k^{(F)}}$ is associative.

Proof. This follows from the fact that the B_{pq} satisfy the identities (1), and from Proposition C.3 with $\alpha = 3$. □

E.2.2. *The maps $x \mapsto x^{(i_1 \ldots i_k)}$.* Let k and l be integers such that $k \leq l$, and let $(i_1, \ldots, i_k)$ be integers in $\{1, \ldots, l\}$ such that $i_1 < i_2 \ldots < i_k$. If $\underline{\alpha} \in \mathbb{N}^k$, define $\underline{\alpha}^{(i_1 \ldots i_k)}$ by $(\underline{\alpha}^{(i_1 \ldots i_k)})_{i_s} = \alpha_s$ and $(\underline{\alpha}^{(i_1 \ldots i_k)})_t = 0$ if $t \notin \{i_1, \ldots, i_k\}$. If $x \in X(\underline{\alpha})$, define $x^{(i_1 \ldots i_k)}$ as the element of $X(\underline{\alpha}^{(i_1 \ldots i_k)})$ equal to the composition of x with the lexicographical bijection between $I_{\underline{\alpha}^{(i_1 \ldots i_k)}}$ and $I_{\underline{\alpha}}$. If $\underline{p} \in P(x, \underline{\alpha})$, define $\underline{p}^{(i_1 \ldots i_l)}$ as the element of $P(x^{(i_1 \ldots i_k)}, \underline{\alpha}^{(i_1 \ldots i_k)})$ given by the compisition of $\underline{p}$ with the square of this bijection.

Let $x \mapsto x^{(i_1 \ldots i_k)}$ be the linear map from $\mathrm{Sh}_k^{(F)}$ to $\mathrm{Sh}_l^{(F)}$ defined as the direct sum of all canonical injections of $\mathrm{Sh}_k^{(F)}(\underline{\alpha}, x, \underline{p})$ into $\mathrm{Sh}_l^{(F)}(\underline{\alpha}^{(i_1 \ldots i_k)}, x^{(i_1 \ldots i_k)}, \underline{p}^{(i_1 \ldots i_k)})$.

Then $x \mapsto x^{(i_1 \ldots i_k)}$ is an algebra morphism, and if $j_1, \ldots, j_{k'}$ are such that $1 \leq j_1 < \ldots < j_{k'} \leq l$ and that $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{k'}\}$ are disjoint, then $[x^{(i_1 \ldots i_k)}, y^{(j_1 \ldots j_{k'})}] = 0$ for any $x \in \mathrm{Sh}_k^{(F)}$ and $y \in \mathrm{Sh}_l^{(F)}$.

E.2.3. *The morphisms $\Delta_{k,i}$.* Assume that k is an integer and let i be an integer such that $1 \leq i \leq k$.

If $\underline{\alpha} \in \mathbb{N}^k$ and $\beta = 1, \ldots, \alpha_i$, define $\underline{\alpha}(i, \beta)$ as the element of $\mathbb{N}^{k+1}$ equal to $(\alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_i - \beta, \alpha_{i+1}, \ldots, \alpha_k)$. If $x \in X(\underline{\alpha})$, define $x(i, \beta) = x'$ as the element of $X(\underline{\alpha}(i, \beta))$ such that $x'(j, \gamma) = x(j, \gamma)$ if $j \leq i$, $x'(j, \gamma) = x(j - 1, \gamma)$ if $j \geq i + 2$, and $x'(i + 1, \gamma) = x(i, \beta + \gamma)$; this is the composition of x with the lexicographical bijection between $I_{\underline{\alpha}}$ and $I_{\underline{\alpha}(i,\beta)}$. If $\underline{p} \in P(x, \underline{\alpha})$, define $\underline{p}(i, \beta)$ as the composition of $\underline{p}$ with the square of this bijection.

Define $\Delta_{k,\underline{\alpha},i,\beta}$ as the canonical injection map from $\mathrm{Sh}_k^{(F)}(\underline{\alpha}, x, \underline{p})$ to

$$\mathrm{Sh}_{k+1}^{(F)}(\underline{\alpha}(i, \beta), x(i, \beta), \underline{p}(i, \beta))$$

and $\Delta_{k,i}$ as $\oplus_{\underline{\alpha} \in \mathbb{N}^k, \beta = 1, \ldots, \alpha_i} \Delta_{k,\underline{\alpha},i,\beta}$. Then $\Delta_{k,i}$ is a linear map from $\mathrm{Sh}_k^{(F)}$ to $\mathrm{Sh}_{k+1}^{(F)}$.

Moreover, $\Delta_{k,i}$ is an algebra morphism, and the maps $\Delta_{k,i}$ and $x \mapsto x^{(i_1 \ldots i_k)}$ satisfy coassociativity and compatibility rules

$$\Delta_{k+1,j} \circ \Delta_{k,i} = \Delta_{k+1,i+1} \circ \Delta_{k,j}$$

if $j \leq i$,

$$\Delta_{l,i}(x^{(i_1 \ldots i_k)}) = x^{(i_1, \ldots, i_s, i_{s+1}+1, \ldots i_k+1)}$$

if $s \neq 0, k$ and $i_s < i < i_{s+1}$, or if $s = 0$ and $i < i_1$, or if $s = k$ and $i > i_k$, and

$$\Delta_{l,i_s}(x^{(i_1 \ldots i_k)}) = (\Delta_{k,s}(x))^{(i_1, \ldots, i_s, i_s+1, \ldots i_k+1)}.$$

E.3. **Universal properties of $\mathrm{Sh}_k^{(F)}$.** Let $(\mathfrak{a}, r_{\mathfrak{a}})$ be the pair of a Lie algebra and a solution $r_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a}$ of CYBE. If k is an integer ≥ 0, $\underline{\alpha} \in \mathbb{N}^k$, $x \in X(\underline{\alpha})$, let us denote by $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}(k, \underline{\alpha}, x)$ the linear map from $\mathrm{Sh}_k^{(F)}$ to $\mathrm{Sh}(\mathfrak{a})^{\otimes k}$ given by the composition of $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{(x(1,1), \ldots, x(k, \alpha_k))}$, the canonical isomorphism $\mathfrak{a}^{\otimes(\alpha_1 + \cdots + \alpha_k)} \rightarrow \otimes_{i=1}^k \mathfrak{a}^{\otimes \alpha_i}$ and the tensor product $\otimes_{i=1}^k \iota_{\alpha_i}$, where ι_s is the canonical injection of $\mathfrak{a}^{\otimes s}$ in $\mathrm{Sh}(\mathfrak{a})$ as its part of degree s.

Define $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\mathrm{Sh}_k^{(F)}}$ as the linear map from $\mathrm{Sh}_k^{(F)}$ to $\mathrm{Sh}(\mathfrak{a})^{\otimes k}$ equal to the sum

$$\sum_{\underline{\alpha} \in \mathbb{N}^k} \sum_{x \in X(\underline{\alpha})} \kappa_{\mathfrak{a}, r_{\mathfrak{a}}}(k, \underline{\alpha}, x).$$

Proposition E.2. $\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\mathrm{Sh}_k^{(F)}}$ *is an algebra morphism. Moreover, we have*

$$\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\mathrm{Sh}_{k+1}^{(F)}} \circ \Delta_{k,i} = \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})}^{\otimes(i-1)} \otimes \Delta_{\mathrm{Sh}(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})}^{\otimes(k-i-1)} \circ \kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\mathrm{Sh}_k^{(F)}}$$

and

$$\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\mathrm{Sh}_l^{(F)}}(x^{(i_1 \ldots i_k)}) = (\kappa_{\mathfrak{a}, r_{\mathfrak{a}}}^{\mathrm{Sh}_k^{(F)}}(x))^{(i_1 \ldots i_k)}$$

for any $x \in \mathrm{Sh}_k^{(F)}$.

The proof is straightforward. This Proposition explains why $\mathrm{Sh}_k^{(F)}, m_k, \Delta_{k,i}$ and $x \mapsto x^{(i_1 \ldots i_k)}$ should be viewed as universal versions of $\mathrm{Sh}(\mathfrak{a})^{\otimes k}, m_{\mathrm{Sh}(\mathfrak{a})}^{\otimes k}, \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})}^{\otimes (i-1)} \otimes \Delta_{\mathrm{Sh}(\mathfrak{a})} \otimes \mathrm{id}_{\mathrm{Sh}(\mathfrak{a})}^{\otimes(k-i-1)}$ and the map $x \mapsto x^{(i_1 \ldots i_k)}$.

E.4. **Proof of Theorem 3.2.** In the statement of Proposition 3.5, we may replace $A^{\otimes k}$ $(k = 2, 3, 4)$ by $\mathrm{Sh}_k^{(F)}$ and r_A by the element $r_{\mathrm{Sh}_2^{(F)}} \in \mathrm{Sh}_2^{(F)}((1,1),(1 \mapsto a, 2 \mapsto b),(1,2) \mapsto 1) \subset \mathrm{Sh}_2^{(F)}$ equal to

$$r \in F_1 = F_1^{(ab)}.$$

The proof is a direct transposition of the proof of Proposition 3.5, which is in Appendix A.

Then using the maps $\Delta_{k,i}$ and $x \mapsto x^{(i_1 \ldots i_k)}$, we may reproduce step by step the proof of Theorem 3.1. This proves Theorem 3.2. $\qquad \square$

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